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MHD WAVE COUPLING IN INHOMOGENEOUS MEDIA

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Abstract. The coupling of different MHD waves is important for an understanding of laboratory, space, and solar plasmas. In this paper we investigate fast and Alfvén mode coupling in the cold plasma limit. The medium is assumed to have an invariant direction (e.g., slab or axisymmetric models) and carries an arbitrary solenoidal background magnetic field. Wave fields are Fourier analysed with wave number k_β in the invariant direction. The coupled linearised wave fields are then expanded in a power series (of k_β) to yield a decoupled hierarchy of inhomogeneous wave equations. The higher order terms in the series represent wave coupling phenomena such as the resonant excitation of Alfvén waves, and the damping of a magnetospheric cavity mode.

Introduction

The basic behaviour of many space plasma phenomena can be understood in terms of MHD waves. For example, long period geomagnetic pulsations have been described in terms resonant Alfvén wave excitation [*Southwood,* 1974; *Chen and Hasegawa,* 1974], and jovian magnetic field fluctuations have been interpreted as both fast and Alfvén modes [*Glassmeier et al.,* 1989]. Detailed modelling of MHD wave phenomena has been retricted in one way or another to date: Some models are analytical, but employ highly idealised media which yield seperable solutions of ordinary differential equations. Other models use realistic magnetic geometries, but impose a time dependence $e^{i\omega t}$ [*Inhester,* 1986; *Chen and Cowley,* 1989; *Mond et al.,* 1990]. The more general studies (with arbitrary time-dependence and/or realistic magnetic field geometry) tend to be numerical [*Allan et al.,* 1987; *Inhester,* 1987; *Lee and Lysak,* 1991].

One direction in which it would be desirable to develop modelling would be to consider time-dependent solutions of the cold plasma equations in general curl-free magnetic fields analytically. Such a calculation is presented briefly in the letter. Naturally, we must impose some simplifications to achieve this goal: We require that the medium is invariant in one direction perpendicular to the background magnetic field (say, $\hat{\boldsymbol{\beta}}$) and that all perturbations have an implicit dependence of $\exp(ik_\beta\beta)$.

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Coupled Linear Equations

The governing equations are linearised and written out in terms of field aligned spatial coordinates (α, β, γ) [*Singer et al.,* 1981; *Southwood and Hughes,* 1983; *Wright and Smith,* 1990]. The curl-free background magnetic field $\mathbf{B}$ lies parallel to the $\hat{\gamma}$ direction everywhere, while $\hat{\boldsymbol{\beta}}$ is aligned with the invariant direction perpendicular to $\mathbf{B}$. The right handed coordinate system is completed by $\hat{\boldsymbol{\alpha}}$. (In an axisymmetric magnetosphere α could be the L-shell parameter.) Such a coordinate system introduces a set of geometrical scale factors $\{h_\alpha, h_\beta, h_\gamma\}$ which describe the magnetic field geometry [*Davis and Snider,* 1979]. A physical interpretation of the factors can be gleaned by realising that a real space element dr is equal to $\hat{\alpha}h_\alpha d\alpha + \hat{\beta}h_\beta d\beta + \hat{\gamma}h_\gamma d\gamma$. The explicit form of these scale factors for a three dimensional diople magnetic field is well known [*Chen and Cowley,* 1989; *Wright,* 1990a; and references therein]. The governing linearised equations may be written, in terms of the perturbation plasma displacement $\hat{\boldsymbol{\alpha}}\xi_\alpha + \hat{\boldsymbol{\beta}}\xi_\beta$

$$D_F(\xi_\alpha) = -ik_\beta \frac{\partial}{\partial\alpha}\left(\frac{h_\gamma B}{h_\beta}\cdot\xi_\beta\right) \quad (1)$$

$$D_A(\xi_\beta) = -ik_\beta\left(\frac{h_\gamma}{h_\alpha h_\beta}\cdot\frac{\partial}{\partial\alpha}(\xi_\alpha h_\beta B)\right) + k_\beta^2\left(\frac{h_\gamma B}{h_\beta}\cdot\xi_\beta\right) \quad (2)$$

The fast mode wave operator D_F and the Alfvén mode wave operator D_A are defined to be

$$\begin{aligned} D_F(\xi_\alpha) \equiv& \frac{\partial}{\partial\gamma}\left(\frac{h_\alpha}{h_\beta h_\gamma}\cdot\frac{\partial}{\partial\gamma}(\xi_\alpha h_\beta B)\right) \\ &+\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma}{h_\alpha h_\beta}\cdot\frac{\partial}{\partial\alpha}(\xi_\alpha h_\beta B)\right) - h_\alpha h_\gamma\frac{B}{V^2}\cdot\frac{\partial^2\xi_\alpha}{\partial t^2} \\ D_A(\xi_\beta) \equiv& \frac{\partial}{\partial\gamma}\left(\frac{h_\beta}{h_\alpha h_\gamma}\cdot\frac{\partial}{\partial\gamma}(\xi_\beta h_\alpha B)\right) - h_\beta h_\gamma\frac{B}{V^2}\cdot\frac{\partial^2\xi_\beta}{\partial t^2} \end{aligned} \quad (3)$$

Evidently, when $\partial/\partial\beta = 0$ (i.e. $k_\beta = 0$) we have decoupled fast waves (the ξ_α wave field) and Alfvén waves (the ξ_β wave field) [*Dungey,* 1954].

The linearised equations above retain the leading order terms (of order ε, say) but neglect the second order terms (of order ε^2). The equations, as they stand above (when $k_\beta \neq 0$), represent a coupled set of equations which are formidable to solve in general. Considerable simplification results if we employ the quantity k_β as a second expansion parameter with which to expand the first order (in ε) perturbations,

$$\xi_\alpha = \xi_\alpha^{(0)} + k_\beta \xi_\alpha^{(1)} + k_\beta^2 \xi_\alpha^{(2)} + ...$$
$$\xi_\beta = \xi_\beta^{(0)} + k_\beta \xi_\beta^{(1)} + k_\beta^2 \xi_\beta^{(2)} + ... \tag{4}$$

(All quantities are dimensionless; both ξ and k_β have been normalised relative to some appropriate background scale length.) The series in (4) can be taken up to an order $n = |\ln(\varepsilon)/\ln(k_\beta)|$ before we need to include the nonlinear terms neglected in the initial linearisation (i.e., where $k_\beta^n \sim \varepsilon$). For the remainder of this paper we shall focus upon the first few terms in this series, and assume that $|\ln(\varepsilon)/\ln(k_\beta)| > 3$.

Decoupled Hierarchy of Equations

The highly coupled system of equations (1) and (2) is transformed by the series solution (4) in to a decoupled hierarchy of equations. Each equation in this hierarchy is of a relatively simple form, namely that of a driven wave equation.

Zeroth Order Solution

The lowest order (in k_β) of equations (1) and (2) are simply the homogeneous wave equations

$$D_F(\xi_\alpha^{(0)}) = 0; \qquad D_A(\xi_\beta^{(0)}) = 0 \tag{5}$$

Thus, to lowest order, the solutions for the fast (ξ_α) and Alfvén (ξ_β) waves are decoupled.

If we are interested in the existence of fast modes in the terrestrial magnetosphere, to lowest order, we simply solve $D_F(\xi_\alpha^{(0)}) = 0$ with the appropriate boundary conditions. For example, a Kelvin-Helmholtz source at the magnetopause would produce a $\xi_\alpha^{(0)}$ that was spatially evanescent and temporally oscillatory. It is also relatively easy to solve for the magnetospheric cavity modes discussed by *Kivelson and Southwood* [1985]. Alternatively, a fast mode could be excited by the collision of a dense plasma cloud with the magnetopause, or the motion of reconnected flux along the magnetopause. Once suitable boundary conditions have been prescribed the homogenous wave equation can be solved.

The lowest order Alfvén solution satisfies the homogenous wave equation $D_A(\xi_\beta^{(0)}) = 0$. Once again it is a matter of defining boundary conditions to permit us to describe a variety of Alfvén waves from those excited by Io [*Smith and Wright,* 1989] to the large scale current system associated with the acceleration of torus plasma [*Glassmeier et al.,* 1989].

Decoupled calculations may provide much insight into the behaviour of a system and can often represent the lowest order solution. However, in most non-trivial systems wave coupling is an inevitable and important property [*Cross,* 1988; *Wright,* 1990a; *Wright,* 1990b; *Wright and Evans,* 1991] and provides the motivation for the present study. We can see below that the first order solutions ($k_\beta\xi_\alpha^{(1)}, k_\beta\xi_\beta^{(1)}$) are a correction to the decoupled solutions, and represent the effect of wave coupling.

First Order Solution

The first order (in k_β) terms of the coupled equations (1) and (2) give us the next two equations in the decoupled hierarchy

$$D_F(\xi_\alpha^{(1)}) = -i\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma B}{h_\beta}\cdot\xi_\beta^{(0)}\right) \tag{6}$$

$$D_A(\xi_\beta^{(1)}) = -i\frac{h_\gamma}{h_\alpha h_\beta}\cdot\frac{\partial}{\partial\alpha}\left(\xi_\alpha^{(0)} h_\beta B\right) \tag{7}$$

The inhomogeneous wave equations above demonstrate clearly how an Alfvén wave ($\xi_\beta^{(0)}$) will couple to a fast mode ($\xi_\alpha^{(1)}$), and how the fast mode ($\xi_\alpha^{(0)}$) will drive an Alfvén wave response ($\xi_\beta^{(1)}$).

The driven Alfvén fields ($\xi_\beta^{(1)}$) could represent a magnetic pulsation. Indeed, our formulation can be shown to exhibit the 'resonant' coupling familiar in simpler geometries. First, we need to consider the normal Alfvén modes of the operator D_A. For perfectly conducting massive ionospheres we have a Sturm-Liouville problem. Each field line (α) has a set of normal modes $\{\xi_{\beta n}(\alpha,\gamma)\}$ which oscillate at their natural frequencies $\{\omega_{\beta n}(\alpha)\}$ respectively. It is well known that on any given field line two different modes are orthogonal [*Morse and Feshbach,* 1953]

$$\int_{\gamma_1(\alpha)}^{\gamma_2(\alpha)} \xi_{\beta n}\xi_{\beta n'} h_\alpha h_\beta h_\gamma \frac{B^2}{V^2} d\gamma = \delta_{nn'} \tag{8}$$

(The ends of the field lines are at $\gamma_1(\alpha)$ and $\gamma_2(\alpha)$.) The modes are also complete, in the sense that we may write an arbitrary disturbance as a sum over the modes weighted with appropriate coefficients $\{a_{\beta n}^{(1)}(\alpha,t)\}$ [*Chen and Cowley,* 1989; *Wright and Smith,* 1990], for example,

$$\xi_\beta^{(1)}(\alpha,\gamma,t) = \sum_n a_{\beta n}^{(1)}(\alpha,t)\xi_{\beta n}(\alpha,\gamma) \tag{9}$$

The evolution of any coefficient ($a_{\beta n}^{(1)}$) can be found by substituting (9) into (7) and employing the orthogonality property (8). (This technique is familiar in quantum mechanics where it is known as 'time-dependent perturbation theory' [*Schiff,* 1968].)

$$\frac{d^2 a_{\beta n}^{(1)}}{dt^2} + \omega_{\beta n}^2 a_{\beta n}^{(1)} = i\int_{\gamma_1(\alpha)}^{\gamma_2(\alpha)} \frac{Bh_\gamma}{h_\beta}\xi_{\beta n}\frac{\partial}{\partial\alpha}\left(h_\beta B\xi_\alpha^{(0)}\right) d\gamma \tag{10}$$

If $\xi_\alpha^{(0)}$ has an oscillatory behaviour (at frequency ω_α), then we may represent the r.h.s of (10) by the driver $C_{no}^{(1)}(\alpha)\exp(i\omega_\alpha t)$. On field lines where ω_α is equal to a natural Alfvén frequency (say, $\omega_{\beta r}(\alpha_r) = \pm\omega_\alpha$) the resonant coefficient grows secularly and is dominated by a term like

$$a_{\beta r}^{(1)}(\alpha_r,t) = \frac{-C_{no}^{(1)}(\alpha_r)t}{2\omega_\alpha}\cdot e^{i\omega_\alpha t} \tag{11}$$

In fact it can be shown that the width of the peak in $a_{\beta r}^{(1)}$ is proportional to $1/t$ so that the resonant response can be approximated as a delta function at the resonant field lines α_r [*Wright,* 1992a]. (cf. the peak in the distribution function during Landau damping.)

At first sight it would appear that the secular nature of the solution in (11) yields an unbounded and badly-behaved solution. This is not the case since the response around a resonant layer is like a delta function and integrates to give a finite energy and a finite driving term in higher order equations [*Wright,* 1992a].

It is noteworthy that the amplitude of the resonant Alfvén fields is proportional to the 'coupling coefficient' $C_{no}^{(1)}$. This coefficient is proportional to the overlap integral in (10) which represents how effectively the fast mode disturbance $\xi_\alpha^{(0)}$ can drive the Alfvén fields $k_\beta a_{\beta n}^{(1)}\xi_{\beta n}$ (cf. *Southwood and Kivelson* [1986]).

Equation (10) may also be solved for the near-resonant field lines close to the sheet $\alpha = \alpha_r$. In this case the familiar off-resonance behaviour is found

$$a_{\beta r}^{(1)}(\alpha,t) = \frac{-C_{no}^{(1)}(\alpha)}{\omega_\alpha^2 - \omega_{\beta r}^2}\cdot e^{i\omega_\alpha t} \tag{12}$$

The resonance becomes infinitely narrow for a steady driver $e^{i\omega_\alpha t}$. The eigenfrequencies $\omega_{\beta r}(\alpha)$ of the resonant and near-resonant modes are assumed to vary smoothly across the resonant layer, and may be expanded according to

$$\omega_{\beta r}^2(\alpha) = \omega_\alpha^2 + (\alpha-\alpha_r)\cdot\left[\omega_{\beta r}^2(\alpha_r)\right]' \tag{13}$$

where the prime denotes differentiation with respect to α. The β component of the magnetic field perturbation around the resonance is given by the time-integrated induction equation,

$$b_\beta = F(\alpha,\gamma)\cdot\frac{e^{i\omega_\alpha t}}{\alpha-\alpha_r} \tag{14}$$

$$F(\alpha,\gamma) = \frac{-C_{no}^{(1)}(\alpha)}{h_\alpha h_\gamma[\omega_{\beta r}^2]'}\cdot\frac{\partial}{\partial\gamma}\left(\xi_{\beta r}h_\alpha B\right) \tag{15}$$

A self-consitent ordering of perturbations requires that $\nabla\cdot\mathbf{b} = 0$ is satisfied to lowest order by the two transverse components of the magnetic field. (cf. *Chen and Cowley* [1989]). Although the b_α component is strictly a second order quantity (in k_β), it may be infered from integrating $\nabla\cdot\mathbf{b} = 0$ across the resonant layer

$$b_\alpha = \frac{-ik_\beta}{h_\beta h_\gamma}\int b_\beta h_\alpha h_\gamma d\alpha \tag{16}$$

Expanding the function $F(\alpha,\gamma)$ as a Taylor series around $\alpha = \alpha_r$, we may integrate (16) to find the leading behaviour of b_α.

$$b_\alpha = ik_\beta e^{i\omega_\alpha t}F(\alpha_r,\gamma)\cdot\ln(\alpha-\alpha_r) \tag{18}$$

It is well-known that 'box' model magnetospheres have a logarithmic singularity under steady excitation. The analysis above shows that this is also true in more realistic geometries, and is a generalisation of earlier calculations.

Second Order Solution

The decoupled hierarchy can be solved as far as is necessary (up to an order $\sim |\ln(\epsilon)/\ln(k_\beta)|$). In this letter we shall only consider one more order. The second order corrections to the fast and Alfvén fields are the solutions of the driven wave equations

$$D_F(\xi_\alpha^{(2)}) = -i\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma B}{h_\beta}\cdot\xi_\beta^{(1)}\right) \tag{19}$$

$$D_A(\xi_\beta^{(2)}) = -i\frac{h_\gamma}{h_\alpha h_\beta}\cdot\frac{\partial}{\partial\alpha}\left(\xi_\alpha^{(1)}h_\beta B\right) + \frac{Bh_\gamma}{h_\beta}\xi_\beta^{(0)} \tag{20}$$

The first order solutions derived in the previous subsection act as drivers in the inhomogeneous wave equations governing the second order solutions.

Discussion

It is interesting to apply our formulation to the damped cavity mode model of magnetic pulsations suggested by *Kivelson and Southwood* [1985], and modelled quantitatively by *Zhu and Kivelson* [1988]. In the cavity mode model the fast mode is damped as it loses energy to the Alfvén resonance. The damping rates have been calculated explicitly in simple geometries by *Zhu and Kivelson* [1988], and are typically two orders of magnitude smaller than the oscillatory frequency.

In our series solution (4), we would represent the (fast) cavity mode to lowest order by $\xi_\alpha^{(0)}$. Employing the usual boundary conditions of perfectly reflecting ionosphere, plasmapause and magnetopause we solve the fast mode equation (5) to find that $\xi_\alpha^{(0)}$ is a fast eigenmode which oscillates at it's natural eigenfrequency. (In general $\xi_\alpha^{(0)}$ will be a sum of several eigenfunctions [*Lee and Lysak,* 1991].) We shall focus upon a single eigenmode here - say, $\xi_{\alpha c}(\alpha,\gamma)$ with natural frequency $\omega_{\alpha c}$. Note that to lowest order there is no damping present ($\xi_\alpha^{(0)} \propto \xi_{\alpha c}\exp(i\omega_{\alpha c}t)$). We shall assume that there are no lowest order Alfvén fields and set $\xi_\beta^{(0)} = 0$.

The principal first order effect of the oscillatory fast mode $\xi_\alpha^{(0)}$ will be to drive an Alfvén resonance on a specific sheet of field lines (α_r) where $\omega_{\beta r}(\alpha_r) = \pm\omega_{\alpha c}$, as described above. We also reasoned that the Alfvén fields could be approximated in the form (cf. *Southwood and Kivelson* [1986], *Wright* [1992a])

$$\xi_\beta^{(1)} \propto \xi_{\beta r}(\alpha_r,\gamma)\exp(i\omega_{\alpha c}t)\cdot\delta(\alpha=\alpha_r) \tag{21}$$

There is no driving term for the first order fast mode $\xi_\alpha^{(1)}$, and accordingly we set $\xi_\alpha^{(1)} = 0$. Note that to first order there is still no evidence of cavity mode damping.

Now consider the second order solutions ($\xi_\alpha^{(2)},\xi_\beta^{(2)}$). Since $\xi_\alpha^{(1)} = \xi_\beta^{(0)} = 0$ there is no driving term in the wave equation for $\xi_\beta^{(2)}$, (20), and we may set $\xi_\beta^{(2)} = 0$. However, the $\xi_\beta^{(1)}$ solution (21) will act as a driver for the $\xi_\alpha^{(2)}$ equation (19). Moreover, since $\xi_\beta^{(1)}$ was excited resonantly it will oscillate at a frequency $\omega_{\alpha c}$ - see (11) and (21). Thus the wave equation (19) will be driven at one of its natural frequencies, and yield a secular solution for $\xi_\alpha^{(2)}$. The second order estimate of the fast mode wave field ($\xi_\alpha = \xi_\alpha^{(1)} + k_\beta^2 \xi_\alpha^{(2)}$) will represent a damped oscillatory fast cavity mode.

Conclusions

We have presented some new techniques for modelling MHD wave coupling in a cold inhomogeneous magnetoplasma. By employing the wavenumber k_β as an expansion parameter we are able to transform the coupled wave equations into a hierarchy of decoupled equations. The formulation has the advantage that we are able to solve the resulting inhomogeneous wave equations in an arbitrary geometry and with unspecified time-dependence. The zeroth order (in k_β) solutions represent decoupled fast and Alfvén solutions. The first order terms introduce corrections that describe wave coupling effects (such as the excitation of resonant Alfvén waves). The second order corrections describe higher order coupling effects, such as the damping of a cavity mode following the excitation of an Alfvén resonance. Future calculations will investigate the Alfvén response (solutions of (7) and (10)) to a variety of fast waves [*Wright*, 1992b], and will also consider the damping of cavity modes in realistic geometries [*Wright*, 1992a].

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