

Finite lifetimes of ideal poloidal Alfvén waves

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Abstract.

Standing second harmonic poloidal Alfvén waves can be excited by drift-bounce resonance with energetic particle populations in the Earth's magnetosphere. Using a cold, ideal, MHD model, we study the temporal evolution of the resulting poloidal Alfvén waves. Imposing an azimuthal dependence of $\exp(i\lambda y)$ in a "box" model of the magnetosphere, we describe poloidal waves, using a large azimuthal wavenumber λ . In homogeneous media, poloidally polarized waves simply oscillate in time. However, if these waves are excited in a nonuniform medium, we find that their polarization rotates from poloidal to toroidal in time. This polarization change is driven by magnetic field gradients which develop as the poloidal wave fields phase mix in time. Asymptotically, all the initial poloidal wave energy is ultimately transferred to a toroidal polarization. On the basis of this phase mixing we define a poloidal lifetime as the time taken for the poloidal and toroidal amplitudes to become equal. We find that the lifetime is given by $\tau = \lambda / (d\omega_A/dx)$. The irreversible change from a poloidal to a toroidal polarization is in agreement with early studies [Radoski, 1974] but contrary to a recent report [Ding *et al.*, 1995]. Our results support the findings of Radoski. Consequently, poloidal Alfvén waves in the Earth's magnetosphere may have a finite poloidally polarized lifetime, after which they become dominantly toroidal, determined by their azimuthal wavenumber and the local natural Alfvén frequency gradient.

1. Introduction

The study of ultralow frequency (ULF) magnetic pulsations in the Earth's magnetosphere continues to attract the attention of space physicists (see the reviews by Southwood and Hughes [1983], Hughes [1983, 1994], and Wright [1994]). ULF Alfvén waves are believed to have a standing nature in the magnetosphere between the ionospheric endpoints of magnetic field lines [Dungey, 1954]. Continuous pulsations of this type, classified as Pc 3-5 (having periods of 10-45 s, 45-150 s, and 150-600 s, respectively), are regularly observed from the ground by both radars (e.g., the Scandinavian Twin Auroral Radar Experiment [Walker *et al.*, 1979], European Incoherent Scatter [Crowley *et al.*, 1987], and Goose Bay [Walker *et al.*, 1992; Ruohoniemi *et al.*, 1991]) and magnetometers [e.g., Samson *et al.*, 1991; Ziesolleck *et al.*, 1993], and on both geostationary [e.g., Hughes *et al.*, 1977; Hughes, 1980; Takahashi

et al., 1984] and elliptically Earth orbiting satellites (e.g., ISEE [Singer *et al.*, 1982; Hughes and Grard, 1984; Cao *et al.*, 1994] and the Active Magnetospheric Particle Tracer Explorers - Charge Composition Explorer (AMPTE-CCE) [Anderson *et al.*, 1990; Takahashi and Anderson, 1992]).

In addition to their classification in terms of periods, these Pc waves have been statistically grouped in terms of their dominant magnetic field components and their polarizations by Anderson *et al.* [1990]. By using the AMPTE-CCE data from $L = 5 - 9$ and over all local times, Anderson *et al.* undertook a statistical study of ULF wave activity in an attempt to link different types of waves with their energy sources. ULF waves may have energy sources which are either internal or external to the magnetosphere.

External sources, such as solar wind driven Kelvin-Helmholtz waves traveling on the magnetopause, can resonantly excite pulsations on field lines deep within the magnetosphere where their frequency matches the local natural Alfvén frequency of a particular field line [Southwood, 1974; Chen and Hasegawa, 1974]. Similarly, sudden solar wind impulses, incident upon the magnetospheric cavity, can excite inward traveling fast mode waves. These waves can reflect and become standing between an outer boundary (possibly the magnetopause) and a turning point within the magnetosphere. These fast cavity waves can then resonantly drive pul-

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sations on closed magnetic field lines where their frequency matches the local Alfvén frequency [Kivelson *et al.*, 1984; Kivelson and Southwood, 1985, 1986; Allan *et al.*, 1986; Wright, 1992; Wright and Rickard, 1995]. Waves resonantly driven by these mechanisms tend to have low azimuthal wave numbers ($m \lesssim 10$) and are dominantly azimuthally polarized.

Waves can also be driven by energy sources internal to the magnetosphere. For example, the drift bounce resonance mechanism can result in energetic particle populations exciting traveling ULF waves with a large azimuthal wavenumber [Southwood, 1976; Southwood and Kivelson, 1981, 1982; Chen and Hasegawa, 1988] as were observed by Hughes *et al.* [1978]. Similarly, plasma instabilities such as the drift-Alfvén-ballooning mode may excite some types of ULF waves in plasmas with warm components [Chen and Hasegawa, 1991; Chan *et al.*, 1994].

Takahashi *et al.* [1990] observed Pc 5 waves with a period of 200 s and examined the modulation of particle fluxes in the presence of the wave. Their data showed field-aligned flux modulations, and these were attributed to the waves having been driven by resonance with ~ 100 -keV ions. Drift-bounce resonance mechanisms are thought to be responsible for exciting high m , second harmonic radially polarized Pc 4-5 waves, which are often observed by satellites, especially during times of quiet magnetic activity [Kokubun *et al.*, 1989; Anderson *et al.*, 1990; Engebretson *et al.*, 1992; Takahashi and Anderson, 1992]. These high m waves are difficult to observe with ground-based magnetometers, as their signals are believed to be screened by the ionosphere [Hughes and Southwood, 1976]; however satellite observations show them to be consistent with the poloidal guided mode discussed by Radoski [1967] and Dungey [1967].

In this paper we consider numerical studies of these radially polarized Pc 4-5 waves (classified as R waves by Kokubun *et al.* [1989]). Satellite studies show them to occur at all local times [Kokubun *et al.*, 1989; Anderson *et al.*, 1990], with a peak occurrence in the afternoon. They typically occur at $L < 7$ on the dayside and at $L > 7$ on the nightside [Anderson *et al.*, 1990], have a longitudinal extent ranging from 1.5 hours to 8 hours [Engebretson *et al.*, 1992]; and are typically localised in L shell, having widths of $\sim 0.2 - 1.6 R_E$ [Singer *et al.*, 1982]. We should point out that storm time Pc 5 waves [Anderson *et al.*, 1990] have characteristics similar to those of these waves. However, they often have a significant compressional component, are believed to be associated with magnetospheric storms and substorms, and are not the subject of this theoretical study.

In low- β plasmas, at quiet times, the drift-bounce resonance mechanism may be responsible for the excitation of essentially Alfvénic second harmonic Pc 4-5 wave disturbances. Consequently, we examine the temporal evolution of high- m guided poloidal Alfvén waves, which may have been driven by this mechanism, by using a cold, ideal, fully compressible plasma in a box MHD model of the magnetosphere. We find that poloidal

Alfvén waves have a finite lifetime during which their dominant polarization is poloidal. In fact, when they are oscillating in an inhomogeneous plasma, they experience a polarization rotation from poloidal to toroidal, and asymptotically in time approach a purely toroidal polarization state in agreement with the calculation of Radoski [1974], who suggested that the poloidal wave fields decay $\propto 1/t$ for large t . This is in contrast to the results of Ding *et al.* [1995], who studied the evolution of poloidal waves in a dipole geometry, and we comment on this discrepancy in the light of our study.

The paper is structured as follows: Section 2 describes the governing equations and numerical model; section 3 outlines the results, while section 4 discusses our results in the context of the Earth's magnetosphere and compares them to those of other studies. Finally, section 5 concludes our paper.

2. Governing Equations and Numerical Model

In order to study the temporal evolution of poloidal Alfvén waves, we choose to adopt a box model for the magnetosphere; see Figure (1) [e.g., Southwood, 1974]. In this model we straighten the magnetic field lines between their ionospheric endpoints, choose a uniform magnetic field lying purely in the \hat{z} direction, and impose an inhomogeneous Alfvén velocity profile in the \hat{x} (radial or L shell) direction by choosing a density profile $\rho(x)$. The \hat{y} direction then completes the triad and represents the azimuthal direction.

We normalize all lengths with respect to the depth of the magnetosphere (L_x), magnetic fields by the ambient (uniform) magnetic field strength (B_0), and densities by the density in the center of the simulation \hat{x} domain. Adopting infinitely conducting ionospheric boundaries at $z = 0, z_0$ imposes vanishing plasma displacements (i.e., $\xi(z = 0, z_0) = 0$). We assume that the density gradients at the radial boundaries representing the edges of

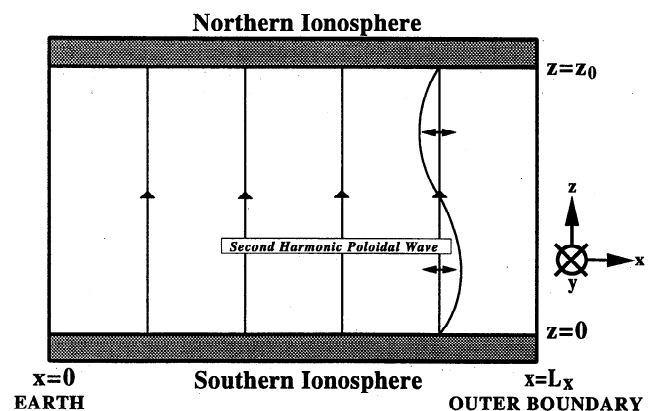


Figure 1. Sketch diagram of the cartesian box model magnetosphere used in this study. Illustrated are schematic plasma displacements associated with a localized second harmonic guided poloidal wave.

the simulation box are sufficiently large that we can apply perfectly reflecting boundary conditions there. We further assume periodic boundary conditions in the \hat{y} direction and hence prescribe the plasma displacements to vary as

$$\xi = (\xi_x(x, t), \xi_y(x, t), 0)e^{i\lambda y} \sin k_z z \quad (1)$$

where $\sin k_z z$ represents the standing nature of the waves between the ionospheric boundaries (often second harmonic for the case of bounce resonance generated poloidal Alfvén waves [see *Southwood and Kivelson, 1982*]) and λ represents the “azimuthal” wavenumber.

Linearizing the ideal MHD equations [e.g., *Boyd and Sanderson, 1969*], and assuming that the plasma is cold, we can describe the temporal evolution of the linear wave fields by the following coupled differential equations (where all variables are normalized):

$$\frac{1}{v_A^2(x)} \frac{\partial^2 \xi_x}{\partial t^2} + k_z^2 \xi_x = -\frac{\partial b_z}{\partial x} \quad (2)$$

$$\frac{1}{v_A^2(x)} \frac{\partial^2 \xi_y}{\partial t^2} + k_z^2 \xi_y = -i\lambda b_z. \quad (3)$$

The time-integrated form of Faraday’s law, combined with ideal hydromagnetic frozen flux condition $\mathbf{E} + \mathbf{v} \wedge \mathbf{B} = 0$, gives

$$b_z = -\left(\frac{\partial \xi_x}{\partial x} + i\lambda \xi_y\right). \quad (4)$$

If $\lambda = 0$ or ∞ , then these equations decouple [*Dungey, 1954, 1967*]. When $\lambda \rightarrow \infty$, $b_z, \xi_y \rightarrow 0$, and ξ_x describes decoupled poloidal Alfvén waves having the general solution

$$\xi_x(x, t) = G(x)e^{i\omega_A(x)t}, \quad (5)$$

where $\omega_A(x)$ is the natural Alfvén frequency of a field line; $\omega_A^2(x) = k_z^2 v_A^2(x)$. This solution describes how the poloidal oscillations of adjacent field lines, which are initially in phase at $t = 0$, drift out of phase with each other in time. Consequently, these initially coherent $\lambda \rightarrow \infty$ poloidal Alfvén waves can be expected to generate fine scales in meridian planes as they phase mix (see *Heyvaerts and Priest [1983]* and *Mann et al. [1995]* for an analysis of the phase mixing of toroidal Alfvén waves).

In order to study the temporal evolution of large (but not infinite) λ poloidal Alfvén waves we solve the above equations, using the generalized matrix eigenvalue method described by *Mann et al. [1995]*. To satisfy the boundary conditions on ξ_x in the \hat{x} direction, we write this perturbation as a half-range Fourier sine series [e.g., *Cally, 1991*],

$$\xi_x = \sum_{m=1}^{\infty} a_m(t) \sin(\pi m x). \quad (6)$$

The form of equations (2), (3), and (4), suggests that ξ_y be expanded in a cosine series,

$$i\xi_y = \frac{1}{2}b_0(t) + \sum_{m=1}^{\infty} b_m(t) \cos(\pi m x). \quad (7)$$

Both these expansions are then truncated at a finite value of $m = N$, so that the resulting finite matrix eigenvalue problem can be solved numerically.

From the prescription of ξ in equation (1), we can see that the real physical displacements ξ_x^P and ξ_y^P vary as $\cos \lambda y$ and $\sin \lambda y$, respectively, and hence that their real physical amplitudes can be extracted from our code as the variables ξ_x and $i\xi_y$. This variation with λ implies that the initial twisted flux tubes comprising the torsional Alfvén waves are standing in the azimuthal direction. Observations and theory show that drift-bounce resonance generated waves are generally propagating in the azimuthal direction [*Southwood and Kivelson, 1982; Chen and Hasegawa, 1988; Takahashi et al., 1990*]. Since traveling waves can be synthesized from a summation of two standing waves, the conclusions from our numerical study should be applicable to the observed traveling waves.

We impose initial wave amplitudes at $t = 0$ and then follow the evolution of the wave fields in time. Calculating the time dependence of the coefficients of the eigenvectors of the inhomogeneous system, corresponding to the imposed initial conditions, allows these coefficients (along with the eigenvectors) to be used to reconstruct the subsequent wave fields at a later time.

3. Results

Throughout this study we impose a monotonically decreasing Alfvén speed profile given by

$$v_A^{-2}(x) = A^2 - B^2 \cos(\pi x) \quad (8)$$

and take $k_z = 1$, $A^2 = 1.0$, $B^2 = 0.1$. We choose $\lambda \gg 1$ to describe poloidal Alfvén waves and vary it as a parameter. We select the initial displacements $\xi_x(x, t = 0)$ and $i\xi_y(x, t = 0)$ so that the initial disturbance represents a torsional Alfvén wave with a purely field guided Poynting flux. This implies that the initial wave disturbance is incompressible, and linearly satisfies $b_z(x, t = 0) = 0$ so that

$$\frac{\partial \xi_x(x, t = 0)}{\partial x} + i\lambda \xi_y(x, t = 0) = 0 \quad (9)$$

(note that increasing λ decreases the initial ξ_y displacement $\propto 1/\lambda$).

Satellite observations of poloidal Alfvén waves show them to be spatially localized in the L shell direction, having widths which are typically $\sim 1-2 R_E$ [*Hughes et al., 1977; Singer et al., 1982; Engebretson et al., 1992*]. Consequently, we choose our initial wave fields to be localized in the \hat{x} direction. Using the Fourier series expansions for ξ_x , we choose the coefficients $a_{m \leq 20}(t = 0)$ to provide a Fourier approximation to the function

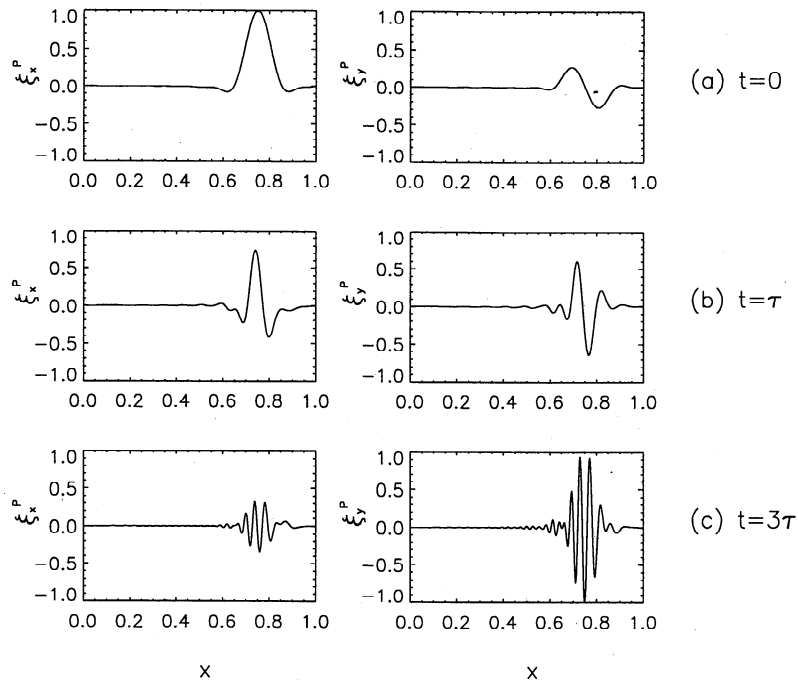


Figure 2. Snapshots of the physical plasma displacements ξ_x^P and ξ_y^P : (a) Initial displacements ($t=0$), (b) $t = \tau$, and (c) $t = 3\tau$.

$$\begin{aligned}
 \xi_x(x, t=0) &= 0 & 0 \leq x < 0.65 \\
 \xi_x(x, t=0) &= 1 - \sin(10\pi x) & 0.65 \leq x \leq 0.85 \\
 \xi_x(x, t=0) &= 0 & 0.85 < x \leq 1.0
 \end{aligned} \tag{10}$$

and the coefficients $b_{m \leq 20}(t=0)$ so that $b_z(x, t=0) = 0$. The remaining coefficients ($21 \leq m \leq N$) are then taken to be zero (N is typically ~ 150). The plasma is initially at rest so that $\dot{a}_m = \dot{b}_m = 0$, where $\dot{a} = da/dt$, and hence the wave energy is initially purely magnetic and stored in the tension of the displaced magnetic field lines. Figure 2 (top row) shows the initial plasma displacements for $\lambda = 50$.

3.1. Temporal Evolution of Poloidal Wave Fields

In Figure 2 (middle and bottom rows) we show the displacements ξ_x^P and ξ_y^P at later times and also for $\lambda = 50$. The phase mixing of the wave fields in both the poloidal and toroidal directions is clearly visible. Also, by the time shown in the middle row the wave is no longer dominated by poloidal motions. In fact, at this time the amplitudes of the two polarizations are approximately equal. We define this to be the time $t = \tau$; see below. By the time shown in the bottom row ($t = 3\tau$) the wave polarizations have switched importance, and the wave is now dominated by toroidal oscillations.

3.2. Toroidal and Poloidal Energy Densities

In order to further examine this polarization rotation we can consider the total energy density of the waves.

The total energy density (e_T) at any point within our simulation box is given by

$$e_T = \frac{1}{2}\rho(\dot{\xi}_x^2 + \dot{\xi}_y^2) + \frac{1}{2\mu_0}(b_x^2 + b_y^2 + b_z^2). \tag{11}$$

We choose to define the energy densities associated with the poloidal (e_p) and the toroidal (e_t) polarizations as

$$e_p = \frac{1}{2}\rho\dot{\xi}_x^2 + \frac{1}{2\mu_0}b_x^2 \tag{12}$$

$$e_t = \frac{1}{2}\rho\dot{\xi}_y^2 + \frac{1}{2\mu_0}b_y^2 \tag{13}$$

and define the compressional energy density as

$$e_c = \frac{1}{2\mu_0}b_z^2. \tag{14}$$

However, e_c is initially zero and remains small compared to e_p and e_t throughout all the high- λ simulations which we consider in this paper, and so it will not concern us further.

In Figure 3 we plot the \hat{x} -integrated energy densities $E_p = \int_0^1 e_p dx$ and $E_t = \int_0^1 e_t dx$ as a function of time for $\lambda = 50$ (on this scale, $E_c = \int_0^1 e_c dx$ is indistinguishable from the time axis). Again the polarization rotation from initially poloidal to asymptotically toroidal is apparent. The natural torsional Alfvén wave period (τ_A) at the center of the initial disturbance (at $x = 0.75$) is 6.5 units. We define the time taken for E_t to equal E_p as the poloidal lifetime τ . Clearly, in this example, $\tau \gg \tau_A$, and we consider this in the next section.

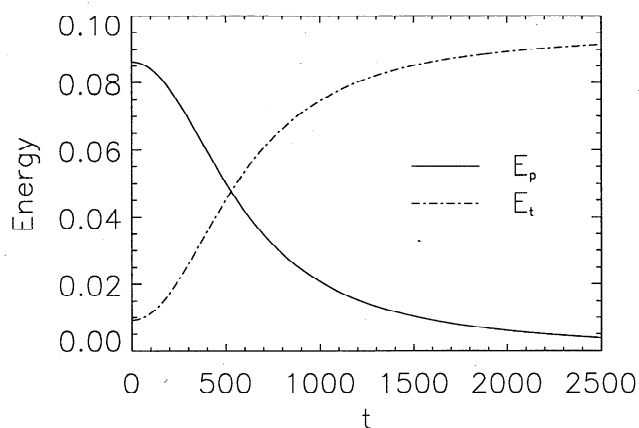


Figure 3. Time evolution of the \hat{x} integrated energy densities $E_p = \int_0^1 e_p dx$ and $E_t = \int_0^1 e_t dx$. The polarization rotation from poloidal to toroidal is clearly shown.

3.3. Poloidal Lifetime as a Function of Azimuthal Wavenumber

In Figure 4 we plot τ (calculated in terms of \hat{x} integrated energy densities, i.e., when $E_p = E_t$) as a function of λ . We have maintained the same initial ξ_x profile and an identical $v_A(x)$. However, in order to satisfy the initial field-guided Poynting flux condition the initial amplitude of ξ_y varies $\propto 1/\lambda$ (see equation (9)). The poloidal lifetime is clearly proportional to λ .

4. Discussion

4.1. Poloidal MHD Alfvén Wave Lifetimes

The numerical results presented in the previous section clearly suggest that large (but finite) λ MHD poloidal Alfvén waves excited in an inhomogeneous plasma will have a finite lifetime with a poloidal polarization. In particular, Figure 4 suggests that asymptotically in time the waves will approach a purely toroidal polarization state.

This concept was first discussed by Radoski [1974]. In his paper he considered how an arbitrary MHD wave disturbance would evolve in time in a cylindrical magnetospheric model comprising a cold, compressible inhomogeneous plasma. By examining the coupled ($\lambda \neq 0$ or ∞) toroidal and poloidal wave equations, Radoski found two apparent resonances; however, only the well known Alfvén resonance (where $\omega = \omega_A(r)$) was found to actually produce singular wave fields.

Since the asymptotic behavior of the wave fields is governed by the singularities in their Fourier transforms, Radoski found that completing a Fourier superposition of the singular terms arising from the Alfvén resonances implied that the poloidal wave fields would decrease $\propto 1/t$, whereas the toroidal wave fields tended (in an ideal plasma) to a nondecaying oscillating function having the same form as the decoupled ($\lambda = 0$) toroidal Alfvén waves.

In our numerical model we introduced initially field-guided torsional (dominantly poloidal) Alfvén waves and observed a similar polarization rotation. Consequently, it appears that not only do low- m compressional poloidal disturbances (fast MHD waves) decay into toroidal waves as they resonantly drive essentially Alfvénic waves at the Alfvén singularity [e.g., Southwood, 1974], but also large- m poloidal waves (themselves essentially incompressible) approach a similar asymptotic state comprising waves with only a toroidal component.

We can further examine this poloidal lifetime by considering the following analysis. Equation (3) implies that with $k_z \sim O(1)$ and $v_A(x) \sim O(1)$, then $\ddot{\xi}_y \sim \xi_y$ and $b_z \sim \xi_y/\lambda$. Thus the left-hand side of (4) is of the order of ξ_y/λ , which must be much less than the last term on the right-hand side when λ is large. To leading order, (4) is the balance of

$$\partial \xi_x / \partial x \sim -i\lambda \xi_y. \quad (15)$$

Thus we find the ordering $\xi_y \sim \xi_x/\lambda$, $b_z \sim \xi_y/\lambda \sim \xi_x/\lambda^2$, when $\lambda \rightarrow \infty$ (strictly we require $\lambda \gg k_z$ and $\partial/\partial x$). In our simulations we set $b_z(x, t=0) = 0$, so that $\partial \xi_x / \partial x|_{t=0} = -i\lambda \xi_y|_{t=0}$ exactly. For cases when $\partial \xi_x / \partial x$ is constant in time (i.e., the background $v_A(x) = \text{const}$), then the initial (dominantly poloidal) torsional Alfvén wave simply oscillates at the local Alfvén frequency, maintaining the displacements ξ_x^P and ξ_y^P in phase in time and keeping the perturbation incompressible (i.e., $b_z(x, t) \sim 0$).

If however $v_A(x) = f(x)$, then to first order we expect ξ_x^P to phase mix according to equation (5). Looking for a solution of the form $\xi_x(t)e^{ik_x x}$, ($k_x = k_x(x, t)$), then, we find that

$$\frac{\partial}{\partial x} \sim ik_x = i\omega'_A(x)t + \frac{G'(x)}{G(x)} \quad (16)$$

and as $t \rightarrow \infty$,

$$k_x \approx \omega'_A(x)t \quad (17)$$

where the prime denotes d/dx .

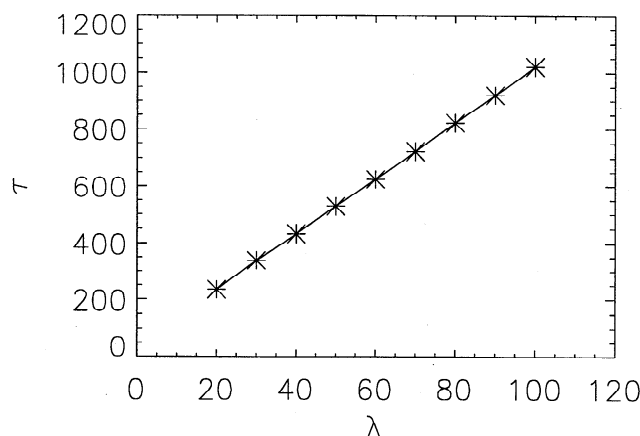


Figure 4. Plot of poloidal lifetime τ versus λ . Asterisks show simulation results using $20 \leq \lambda \leq 100$ (Note that for $\lambda \lesssim 10$ the wave is no longer initially dominantly poloidal.)

Equation (17) can be used to define the radial scale length of an initially uniform disturbance which will be generated at any time via phase mixing,

$$L_{ph} = \frac{2\pi}{k_x} \approx 2\pi(\omega'_A t)^{-1} \quad (18)$$

(see, for example, Mann *et al.* [1995]). Knowing that $b_z \sim 0$ for large λ , then we get from equations (15) and (17)

$$\left| \frac{\xi_x}{\xi_y} \right| \sim \frac{\lambda}{\omega'_A t}. \quad (19)$$

We can interpret this equation in terms of phase mixing. Once the initial poloidal disturbance has phase mixed to a radial scale length $2\pi/\lambda$ (i.e., equal to the azimuthal scale length), then we expect the toroidal amplitude to have grown to equal the poloidal one. This then defines the poloidal lifetime τ ,

$$\tau = \frac{\lambda}{\omega'_A}. \quad (20)$$

Looking again at Figure 4 (based on \hat{x} integrated energy densities), we would expect the gradient of this graph to be given by $(\omega'_A)^{-1}$. For $A^2 = 1.0$, $B^2 = 0.1$, and $k_z = 1$, then at $x = 0.75$ we find that $(\omega'_A)^{-1} = 9.97$. This is in good agreement with the gradient in Figure 4 of 9.87.

We have also investigated cases in which the initial perturbation isn't purely field guided. If we consider the wave field solenoidal constraint, $\nabla \cdot \mathbf{b} = 0$, then we expect

$$k_x b_x + k_y b_y + k_z b_z = 0. \quad (21)$$

In the magnetosphere, $k_z \approx n\pi/L_z$, where L_z is the field line length. For low field-aligned harmonics, we find $k_z \sim 1/L_z \ll \lambda$, normally. Hence with k_x given by equation (17), b_z can be nonzero and still satisfy $k_x b_x \ll k_x b_x, k_y b_y$. Hence the polarization rotation from $b_x \rightarrow b_y$ (or $\xi_x \rightarrow \xi_y$) will still occur as a result of $k_x b_x \sim -k_y b_y$ since k_x steadily increases according to equation (17).

This was verified by our numerical simulations (not shown) and is encouraging since it implies that so long as $k_z \ll \lambda$ (as is the case for these waves in the magnetosphere), the polarization rotation previously discussed will be a robust feature of the wave evolution and does not require a stringent field-guided energy condition.

We note that although b_z is small compared to ξ_x and ξ_y (or b_x and b_y) it is the small b_z that is responsible for coupling the ξ_x and ξ_y equations (2) and (3) and causes the drift from radial to azimuthal polarization. Although b_z is a small perturbation, it is the gradient λb_z or $\partial b_z / \partial x$ that is important, and this will be of a higher order than b_z .

4.2. Magnetospheric Poloidal Alfvén Wave Evolution

The numerical model which we have used in our study was a simple box model comprising straight field lines,

and it allowed a one-dimensional radial time dependent wave solution to be determined. In more complex geometries, such as dipole magnetic fields [Radoski, 1967; Dungey, 1967], the ordering of the field aligned magnetic field component $b_{\parallel} \sim \xi_y/\lambda$ is still valid. Radoski [1967] considered the case of $m \gg 1$ and obtained a guided poloidal mode where

$$\frac{\partial h_L E_L}{\partial \phi} = \frac{\partial h_\phi E_\phi}{\partial L}, \quad (22)$$

where L represents the radial and ϕ the azimuthal coordinates in the dipole geometry, with h_L and h_ϕ scale factors given by $h_L = a \cos^3 \theta / (1 + 3 \sin^2 \theta)^{1/2}$ and $h_\phi = La \cos^3 \theta$. The field lines are described by $r = La \cos^2 \theta$ (a is the Earth's radius (R_E) and θ is dipole latitude). In an entirely analogous way to the straight field case we can expect that as phase mixing occurs in an initially poloidal disturbance, the wave polarization will rotate from poloidal to toroidal so as to keep b_{\parallel} small.

If we adopt the dipole magnetic field toroidal Alfvén wave periods considered by Anderson *et al.* [1989], (neglecting variations with local time), then

$$\omega_A(L) = \frac{CL^{-a}}{(L-1)^{1/2}} \quad (23)$$

where L is the McIlwain parameter, the equatorial plasma density varies as L^{-q} , $a = (7-q)/2$, and C is a constant. When we choose $q = 4$, a typical L value of $L = 7$, and waves with 100 s period and $m = 100$, then (at the equator) $\tau \sim 760 \text{ s} \sim 13 \text{ mins}$. This provides an estimate of the time before ideal poloidal pulsations become dominantly toroidal.

In the real magnetosphere, both the poloidal and toroidal wave components will be damped by driving Pedersen currents in the resistive ionospheric boundaries. A numerical calculation of the ionospheric damping rate of second harmonic poloidal pulsations in a dipole geometry was completed by Newton *et al.*, [1978]. They found that for typical dayside height integrated Pedersen conductivities, $\Sigma_P \sim 10\text{--}20 \text{ mhos}$ ($9 \times 10^{12} - 1.8 \times 10^{13} \text{ esu}$), at $L = 7$, the damping decrement $\gamma/\omega_r \sim 0.005\text{--}0.003$ (see, for example, the top of their Figure 5, which corresponds to fixed end second harmonic poloidal waves). If we define the e -fold ionospheric damping time as τ_I , then these values correspond to $\tau_I = 32\text{--}53$ Alfvén wave periods. With $\tau_A \sim 100 \text{ s}$, then, this gives $\tau_I \sim 0.9\text{--}1.5$ hours. This is much more than the previously estimated poloidal lifetime. Consequently, depending on the ratio τ_I/τ , magnetospheric poloidal waves could experience a polarization rotation from $\xi_x \rightarrow \xi_y$ within their ionospheric lifetimes.

In a realistic kinetic plasma, after a time τ_k the phase mixing will reach a radial length scale $L_k = 2\pi/\omega'_A \tau_k$, such as the ion Larmor radius. At this point the phase mixing may saturate at radial scales of $\sim L_k$ [e.g., Rankin *et al.*, 1993], and hence the MHD results will require revision. However, if $\tau < \tau_k$, then we can predict that waves which are observed to have a dominant

poloidal polarization will have been driven a time less than τ ago.

Local plasma conditions may be critical in determining when high- m second harmonic poloidal waves are excited. *Engbreton et al.* [1992] found by using AMPTE-CCE data that Pc 4 waves of this type are preferentially excited during quiet geomagnetic conditions and are perhaps associated with plasmaspheric refilling, which creates conditions favorable for wave excitation. *Singer et al.* [1982] also observed Pc4 waves of this type which were correlated with plasmopause density gradients. They also saw a wave gap in their data between regions supporting pulsations, associated with a plasma density depletion, which suggests that the ambient plasma density might be critical in determining where waves are driven. If local plasma conditions are important, then these same local conditions could determine the speed with which the polarization rotation occurs. In regions where ω'_A is large (where waves are often observed), the poloidal Alfvén wave lifetime will be small, and thus these waves will only be dominantly poloidally polarized for a short time.

Our simulations have considered the solution to an initial value problem: The boundary conditions correspond to the initial bounce resonance driven fields. In that sense, our results may be more applicable to situations in which the waves are no longer being driven by the energetic particles. Where a continuous energy source is present, and the wave particle interactions remain active, a summation over repeatedly driven evolving wave solutions may be a more accurate representation of the behavior which would be observed.

We should also note that in a dipole magnetic field the eigenfrequencies of the undriven, decoupled poloidal and toroidal wave operators are different because of the different functional variation of the scale factors in each of the operators. The frequency variation we have used above strictly applies to toroidal rather than poloidal waves. However, since the difference between the two eigenfrequencies is typically small (for example *Wright and Smith* [1990] found a <1% difference between the poloidal and toroidal eigenfrequencies for second and higher harmonics in a study of Alfvén waves in the Jovian magnetosphere), their variation with L should be similar, and the above estimate of τ should be a reasonable approximation.

The fact that the eigenfrequencies of the two operators are different on a given field line makes it technically impossible to synthesize a purely torsional Alfvén wave in a geometry such as a dipole [*Cross*, 1988]. In fact, only very specific geometries (such as the box model) allow purely torsional waves to be generated [*Wright*, 1990]. This is a point to which we return in section 4.4.

4.3. Hodograms of Poloidal MHD Alfvén Wave Fields

In addition to considering simply the polarization rotation experienced by evolving poloidal waves, we can also examine the evolution which would be observed by satellites crossing them. In Figure 5 we plot hodograms

showing the evolution of the physical displacements ξ_x^P and ξ_y^P , each panel showing a time interval of $3\tau_A$ (we choose $\lambda = 50$ so that $\tau \sim 77\tau_A$). The columns show the fields at the center and on either side of the initial disturbance profile (at $x = 0.7, 0.75, 0.8$), while the rows show the waves at different times during the simulation.

We can clearly see at $t = 0$ that ξ_x^P dominates ξ_y^P , and initially these disturbances oscillate in phase (or antiphase, depending on their position relative to $x = 0.75$) and represent azimuthally standing torsional Alfvén waves having field-guided Poynting flux. However, by $t = \tau$, the amplitude $\xi_x^P \approx \xi_y^P$, as discussed in section 4.1. At this time we can see that the component of ξ_y^P which has been driven by phase mixing is now $\pi/2$ out of phase with ξ_x^P (this is particularly clear in the center panel). This is to be expected from the analysis in section 4.1, since the driven component of $\xi_y^P = -ik_x \xi_x^P / \lambda$, with k_x as given in equation (17). Instead of being linearly polarized the waves have become circularly polarized at $t = \tau$. As t increases further, ξ_x^P continues to decrease in amplitude, and the waves develop an elliptical polarization. At $x = 0.7, 0.8$, the polarization ellipses are slightly skewed, owing to the initial ξ_y^P , which was imposed to ensure $b_z(x, t = 0) = 0$.

Azimuthally traveling solutions may be generated by summing two standing wave solutions, one of which has been presented here. The change from poloidal to toroidal polarization is preserved; however, at $t = \tau$ the traveling waves are more linearly than circularly polarized.

Hughes and Grard [1984] used data from GEOS 1, ISEE 1 and ISEE 2 to study second harmonic geomagnetic pulsations which were observed at the inner boundary of the plasma sheet. These waves were dominantly radially polarized, and were believed to have been driven by bounce resonance with energetic (~ 5 keV) ions. The polarizations of the waves observed by the ISEE satellites were unusual. In Figure 8 of *Hughes and Grard*, [1984] both ISEE 1 and ISEE 2 saw elliptically polarized waves whose transverse field amplitudes varied with UT. However, the polarization changes seen by the two satellites could not be explained simply in terms of just static LT or coincident UT descriptions. We do not attempt to reanalyze the data of Hughes and Grard here, but we believe that the polarization rotation we have discussed could be important in the evolution of waves such as those which they observed. If poloidal waves do experience a polarization rotation in time, then the ellipticity and the skewed angle of the polarization ellipse at any time will be determined both by the details of the drift-bounce resonance excitation mechanism (which will govern the initial wave configuration) and by the evolution of the wave fields in the inhomogeneous magnetospheric plasma. Consequently, the time at which a satellite such as ISEE crosses a pulsation (which will be localized in L shell) may be critical in determining the wave fields which are observed.

In addition, poloidal Alfvén waves are often observed [*Takahashi et al.*, 1990] to be propagating in the azimuthal direction (a requirement if they are driven by a drift or drift-bounce resonance mechanism [*Southwood*

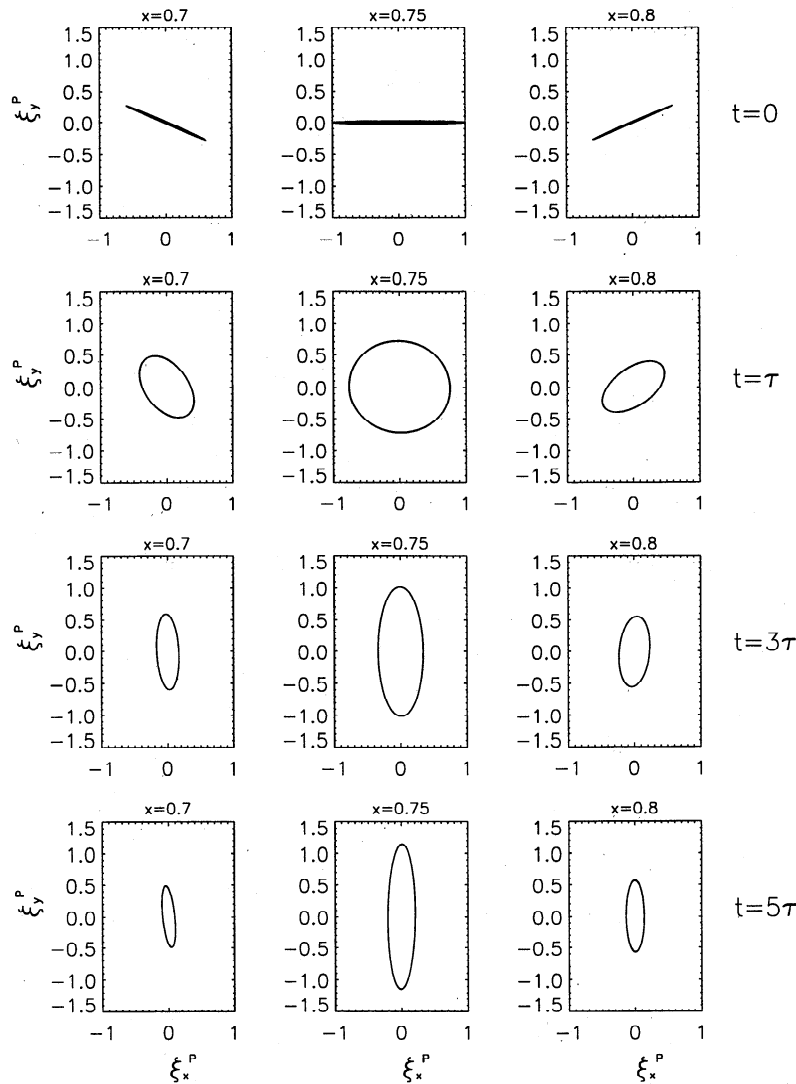


Figure 5. Hodograms showing the evolution of the plasma displacements ξ_y^P versus ξ_x^P . Each panel shows three Alfvén periods of poloidal wave evolution. Columns depict the wave variation in L at $x = 0.7$, $x = 0.75$ (at the center of the initial disturbance), and $x = 0.8$. Rows show the fields at times $t = 0$, $t = \tau$, $t = 3\tau$, and $t = 5\tau$.

and Kivelson, 1981, 1982]). Hence the evolving poloidal wave fields can also be moving with respect to an observing satellite. In the ISEE 1 and ISEE 2 observations of Hughes and Grard [1984] (at $L \sim 7-8$) the two satellites observed poloidal waves with opposite polarizations for $t \sim 30$ mins, although they were separated by only ~ 9 mins in their orbits. This was interpreted as possibly indicating that the poloidal wave disturbances were following the orbiting ISEE satellites. Our estimate of τ , for similar L shells, from the previous section was only 13 mins. Consequently, in analyses of data from poloidal Alfvén waves we believe that the polarization rotation we have discussed should be included because it constitutes an important feature of high- m poloidal wave evolution in the magnetosphere.

4.4. Dipolar Poloidal Alfvén Wave Studies

Recently, Ding *et al.* [1995] considered the temporal evolution of poloidal Alfvén waves in a dipole geome-

try. Some of their results were quite different in character from those found by Radoski [1974] and our present study. At early times the results of Ding *et al.* [1995, their Figure 2] agree with previous work and show a switch from poloidal to toroidal perturbations. However, at later times the polarization changes from being toroidal to poloidal to toroidal in a cyclic pattern. This behavior is at odds with the irreversible transition from poloidal to toroidal motions discussed in other studies.

Why is there a discrepancy? Given the grid resolution and other quantities from Ding *et al.*'s paper, we anticipate that the phase mixing length will equal the numerical grid resolution size after a time $t = 25$ (from equation (18)). For $t > 25$ the phase mixing scale of the waves can not be resolved by the grid, and this is the reason for the unusual polarization behavior observed in this interval. For $t < 25$ the scheme should resolve the wave fields, and the results reported for $0 < t < 25$ are in agreement with those of Radoski [1974] and our solutions.

Indeed, these ideas are borne out by a comparison run reported by Ding et al. which employed increased spatial resolution: From (18) we would expect these results to be an accurate solution over a longer period than 25, and this is in accord with their discussion as well as the results from other unpublished simulations. It may be possible that if the scale of waves in nature is limited somehow (perhaps by kinetic effects), then the irreversible change to toroidal motions may be arrested. Ding et al.'s results could represent one of the possible long-term solutions (R. E. Denton, personal communication, 1995).

With the above provisos, it appears that ideal MHD poloidal Alfvén waves will experience a polarization rotation from poloidal to toroidal in box (present study), hemicylindrical [Radoski, 1974] and even dipole [Ding et al., 1995] geometries. Thus it is likely that this behavior will be observed in real magnetospheric geometries.

5. Conclusions

We have presented the results of a cold, ideal, linear, MHD study of the temporal evolution of poloidal Alfvén waves in an inhomogeneous plasma. Using a one-dimensional box model for the magnetosphere, we numerically investigate the evolution of the coupled poloidal (ξ_x) and toroidal (ξ_y) plasma displacements of these waves for large (but noninfinite) azimuthal wavenumbers (λ).

We find that initially poloidally polarized waves, with a field-guided Poynting flux, evolve so as to remain essentially incompressible. We observe that their polarization rotates from a poloidal configuration to asymptotically approach a purely toroidal polarization. We suggest that this is a result of the phase mixing of the initial poloidal disturbances in meridian planes. On this basis we define a poloidal lifetime $\tau \sim \lambda/\omega'_A(x)$, where the prime denotes d/dx . This implies that satellites crossing high- m poloidal Alfvén waves (often observed as Pc 4 waves in the afternoon) will see a different polarization depending on the time since the waves were excited. Moreover, if m and $\omega_A(L)$ are known, it enables us to put limits upon when a poloidal Alfvén wave was excited.

In conclusion, we believe that high- m poloidal Alfvén waves will initially evolve as decoupled field line oscillations which phase mix in meridian planes. Asymptotically in time, their poloidal polarization will become toroidal, as first envisaged by Radoski [1974], and the waves will become decoupled toroidal field line oscillations. The dominance of decoupled field line oscillations in the evolution of waves in inhomogeneous plasmas has been previously discussed in the literature [Cally, 1991; Cally and Sedláček, 1994; Mann et al., 1995], and we believe that initially field guided poloidal Alfvén waves also reveal this dominant behavior as they evolve in time.

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