

Resonant Alfvén Wave Excitation in Two-Dimensional Systems: Singularities in Partial Differential Equations

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The resonant excitation of Alfvén waves is considered in a cold plasma embedded in a uniform magnetic field $B\hat{z}$. All wave fields are assumed to vary as $\exp[i(\lambda y - \omega t)]$, and the background medium is invariant in y . The background density distribution $\rho_0(x, z)$ is otherwise completely arbitrary. Regular and singular solutions for the waves are derived systematically in the vicinity of a resonance by considering a generalized Frobenius series, and we are able to recover many results found in earlier studies. Some new features of our work include a generalization of the overlap integral determining the efficiency with which any resonance may be excited, and the possibility that $\xi_y \propto 1/x^N$ ($N = 1, 2, 3, \dots$) at the resonance. Hitherto only the solution with $N = 1$ has been considered.

1. INTRODUCTION

The coupling of different MHD wave modes is of interest to many plasma physicists. The resonant coupling of fast and Alfvén waves is an important mechanism for heating laboratory plasmas [Hasegawa and Chen, 1976] and could play a significant role in heating the Sun's chromosphere. In a magnetospheric context it is thought that magnetic pulsations may be understood in terms of resonant wave coupling [Southwood, 1974; Chen and Hasegawa, 1974a].

Over recent years considerable efforts have been made to improve our understanding and modelling of coupling phenomena. Early studies concentrated upon steadily driven one dimensional systems [Southwood, 1974; Chen and Hasegawa, 1974a] and required the treatment of resonant singularities in ordinary differential equations. These singularities occur at the location of the resonant field line. (The solution at the singularity can always be derived in the form of a series by employing the method of Frobenius [e.g., Bender and Orszag, 1978].)

More recent studies have tried to generalize the early results by considering a two-dimensional medium [Inhester, 1986; Southwood and Kivelson, 1986; Lee and Lysak, 1990; Mond et al., 1990; Wright, 1991; Wright, 1992a]. Some of the recent progress has been due to numerical solutions. In this paper we develop analytical methods more fully. Our aims are similar to those of Mond et al., [1990]. In their paper they adapted the series solution approach that had proved successful in early modelling. The more general medium considered by Mond et al., [1990] requires the treatment of resonant field lines in two spatial dimensions rather than one. The resulting equations are partial, rather than ordinary, differential equations (see also Pao [1975]).

In contrast to the ordinary differential equation (ODE) models, there is no standard method for deriving the solution of partial differential equations (PDEs) at a resonance

(i.e., singularity). Mond et al., [1990] investigated a generalized series solution approach for the latter problem. Their calculation employed much physical insight, gained from earlier studies, to order the magnitudes of various perturbation quantities and enabled them to deduce the leading-order behavior of the singular solution - cf. Chen and Cowley [1989]. (Second-order resonance problems have two solutions, one regular and one singular.) General boundary conditions will produce a solution that can always be expressed as a sum of the regular and singular solutions. Thus the singular solution alone can not describe the details of resonance problems adequately.

The analysis presented in this paper provides a systematic procedure which will generate the singular and regular series solutions in the vicinity of the resonance. Moreover, we can achieve these results without recourse to invoking specific orderings of the magnitudes of perturbation quantities *a priori*. In addition, our approach does not require any restriction on the wave number [Wright, 1992a] nor do we require the density variation to be a separable function [e.g., Southwood and Kivelson, 1986].

Most recently, Schulze-Berge et al. [1992] have considered the existence of resonant surfaces within a uniform magnetic field permeating a three-dimensional density variation. Their calculation does, however, assume a given ordering of the perturbations.

The paper is structured as follows; In section 2 we derive the governing equations for a two-dimensional resonance problem in a uniform magnetic field (note that the density distribution varies in two directions). This system is identical to that considered in the "zero pressure" section of the paper by Mond et al. [1990], though we tend to follow the notation of, for example, Southwood [1974], Southwood and Kivelson [1986]. Section 3 derives a generalized Frobenius solution across the resonant field line. Section 4 demonstrates how the logarithmic terms in the Frobenius series may be continued across the resonant layer. Section 5 compares our results with those of previous studies, and section 6 concludes the paper.

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2. GOVERNING EQUATIONS

Throughout this paper we employ the linearised ideal MHD equations for a cold plasma in a uniform magnetic field $\mathbf{B} = B\hat{z}$. The equilibrium plasma density $\rho_0(x, z)$ is invariant in the $\hat{y}$ direction, but otherwise general. We assume that all perturbations have a dependence on y and time t of the form $\exp[i(\lambda y - \omega t)]$, the dependence upon (x, z) is to be determined.

We shall introduce the Alfvén wave operator $\mathcal{L}$, which is defined as the differential operator

$$\mathcal{L} \equiv B^2 \frac{\partial^2}{\partial z^2} + \mu_0 \rho_0(x, z) \omega^2. \quad (1)$$

The momentum and induction equations may be combined with Ampère's and Ohm's laws to yield the following equations governing the transverse plasma displacement (ξ_x, ξ_y) and the compressional field component b_z ,

$$\mathcal{L}\xi_x = B \frac{\partial b_z}{\partial x} \quad (2a)$$

$$\mathcal{L}\xi_y = i\lambda B b_z \quad (2b)$$

$$\frac{b_z}{B} = -\frac{\partial \xi_x}{\partial x} - i\lambda \xi_y. \quad (2c)$$

In earlier studies, where the density was independent of z , it was possible to derive a single ODE for b_z and then use the conventional Frobenius treatment of the resonance. That is, the solution was developed as a power series in x , with the inclusion of a suitable logarithmic term to treat the singularity at the resonant field line [Bender and Orszag, 1978]. This is not possible in our more general model, however, we can simplify matters slightly by eliminating b_z from the above equations

$$\mathcal{L}\xi_x = \frac{1}{i\lambda} \frac{\partial \mathcal{L}\xi_y}{\partial x} \quad (3a)$$

$$\mathcal{L}\xi_y = -i\lambda B^2 \frac{\partial \xi_x}{\partial x} + \lambda^2 B^2 \xi_y. \quad (3b)$$

We shall find it useful to expand the Alfvén wave operator $\mathcal{L}$ as a Taylor series in the x coordinate [cf. Mond et al., 1990]. Without loss of generality we choose $x = 0$ to be the x -coordinate about which the expansion is performed. (The origin of x can always be redefined to study the solution on different field lines.)

$$\mathcal{L} = \mathcal{L}_0 + x\mathcal{L}_1 + x^2\mathcal{L}_2 + \dots \quad (4)$$

The operator $\mathcal{L}_0$ and the functions $\mathcal{L}_1, \mathcal{L}_2, \dots$ are defined as

$$\begin{aligned} \mathcal{L}_0(\omega) &= B^2 \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \rho_0(0, z) \\ \mathcal{L}_1(\omega) &= \mu_0 \omega^2 \left[\frac{\partial \rho_0}{\partial x} \right]_{x=0} \\ \mathcal{L}_n(\omega) &= \mu_0 \frac{\omega^2}{n!} \left[\frac{\partial^n \rho_0}{\partial x^n} \right]_{x=0} \end{aligned} \quad (5)$$

The operator $\mathcal{L}_0$ is the Alfvén wave operator for the field line located at $x = 0$. For the remainder of this paper we shall assume that the magnetic field lines are frozen in to dense perfectly conducting boundaries at $z = 0$ and $z = \ell$, so that ξ_x, ξ_y and b_z all vanish at $z = 0, \ell$. This is the "box model" of Southwood [1974] but with density having a general dependence upon x and z . Under these boundary conditions the wave operator is of the Sturm-Liouville form, and any solution to the operator may be written as a

sum over the eigenmodes (or "harmonics") of the resonant field line. We shall denote the n^{th} Alfvén wave eigenmode by the function $\phi_n(0, z)$, with $n = 1$ corresponding to the fundamental mode. The first argument of $\phi_n(0, z)$ denotes that this is an eigenmode of the field line at $x = 0$: we shall require eigenmodes of other field lines in section 4. (Unless stated explicitly, ϕ_n refers to the eigenmodes at $x = 0$, e.g., in sections 2, 3 and the appendices.) The eigenmodes and their associated eigenfrequencies $\omega_n(x = 0)$ satisfy the relation

$$B^2 \frac{\partial^2 \phi_n}{\partial z^2} + \omega_n^2 \mu_0 \rho_0(0, z) \phi_n = 0; \quad (6)$$

and it is simple to calculate the effect of operating upon any eigenfunction with $\mathcal{L}_0(\omega)$:

$$\mathcal{L}_0(\omega) \phi_n(0, z) = \mu_0 \rho_0(0, z) (\omega^2 - \omega_n^2) \phi_n(0, z). \quad (7)$$

The condition for resonance may be regarded as being that a field line is driven at one of its natural eigenfrequencies, say the r th eigenfrequency ω_r . Then the resonant eigenfunction ϕ_r will satisfy $\mathcal{L}_0(\omega_r) \phi_r(0, z) = 0$.

Because the eigenfunctions ϕ_n form a complete orthogonal set, any function $f(z)$ may be written as a sum over these functions weighted with appropriate coefficient f_n :

$$f(z) = \sum_{n=0}^{\infty} f_n \phi_n(0, z). \quad (8)$$

The eigenmodes are orthogonal in the sense that when the product of two different modes is weighted by the density and integrated in z along the resonant field line the result is zero,

$$\langle \rho_0 \phi_n \phi_{n'} \rangle \equiv \int_{(0,0)}^{(0,\ell)} \rho_0 \phi_n \phi_{n'} dz = 0, \quad n \neq n'. \quad (9)$$

In the following sections we shall see that it is often necessary to invert the operator $\mathcal{L}_0$. For example, suppose we need to solve $\mathcal{L}_0 f(z) = g(z)$ for $f(z)$ when $g(z)$ is a known function. Writing f as the sum in (8), the inversion problem amounts to determining the set of coefficients $\{f_n\}$. Employing (7) and (8), $\mathcal{L}_0 f(z) = g(z)$ may be written

$$\mu_0 \rho_0(0, z) \sum_{n=0}^{\infty} (\omega^2 - \omega_n^2) \phi_n f_n = g(z). \quad (10)$$

The coefficient of any mode may be isolated by multiplying (10) by the relevant mode, and integrating along the $x = 0$ field line. Recalling the orthogonal property (9) we find

$$\mu_0 (\omega^2 - \omega_n^2) \langle \rho_0(0, z) \phi_n \phi_n \rangle f_n = \langle g(z) \phi_n \rangle \quad (11)$$

and hence, provided $\omega_n \neq \omega$,

$$f_n = \frac{\langle g(z) \phi_n \rangle}{\mu_0 (\omega^2 - \omega_n^2) \langle \rho_0(0, z) \phi_n \phi_n \rangle}. \quad (12)$$

This defines the solution f of $\mathcal{L}_0 f = g$, and hence the inverse operator $\mathcal{L}_0^{-1}$, in the case when ω is not equal to any of the eigenfrequencies of the $x = 0$ field line. In this paper, however, we are interested principally in the resonant case, when the driving frequency ω is equal to ω_r for some r . In this case, it is evident from (11) that f_r cannot be determined, although all of the other coefficients are still defined by (12). Indeed, when considering the resonant coefficient ($n \equiv r$) we see that the function g cannot be completely

arbitrary if a solution f is to exist (see equation (11)): a solution f of $\mathcal{L}_0 f(z) = g(z)$ only exists if $g(z)$ satisfies the solvability condition [SC]

$$\langle g(z) \phi_r(0, z) \rangle = 0. \quad (13)$$

The solution $f(z)$ obtained by inverting $\mathcal{L}_0$ has an unspecified amount of the resonant eigenfunction present. Thus the inversion of $\mathcal{L}_0 f(z) = g(z)$, where g satisfies the SC (13) and ω is equal to the natural frequency $\omega_r(0)$, yields

$$f(z) = \sum_{n \neq r} \frac{\langle g(z) \phi_n \rangle}{(\omega_r^2 - \omega_n^2) \langle \rho_0(0, z) \phi_n \phi_n \rangle} \phi_n(0, z) + \gamma \phi_r, \quad (14)$$

where γ is an arbitrary constant. We shall write the right-hand side of equation (14) as being identically equal to

$$\mathcal{L}_0^{-1} g + \gamma \phi_r, \quad (15)$$

thus defining the operator $\mathcal{L}_0^{-1}$ in the resonant case.

3. GENERALIZED FROBENIUS SOLUTION

In this section we outline the generalized Frobenius procedure for solving equations (3). To illustrate the method we develop the nonresonant solution in some detail and then indicate the main differences in the method when the $x = 0$ field line is driven at a resonant frequency. Full details are deferred to Appendices A and B.

Having expanded the operator $\mathcal{L}$ (equation [5]) as a series in x we seek solutions ξ_x and ξ_y to equations (3) by similarly writing these functions as series expansions in x . Guided by the Frobenius solution to ordinary differential equations, we allow for the possibility of a logarithmic term and therefore posit the following ansatz

$$\xi_x = x^\sigma \sum_{n=0}^{\infty} a_n(z) x^n + x^\sigma \ln x \sum_{n=0}^{\infty} c_n(z) x^n \quad (16a)$$

$$\xi_y = x^\sigma \sum_{n=0}^{\infty} b_n(z) x^n + x^\sigma \ln x \sum_{n=0}^{\infty} d_n(z) x^n. \quad (16b)$$

Here the a_n , b_n , c_n and d_n , which are to be determined, are functions of z , in contrast with the ODE case where they would be constants. In order to make σ (which thus far is also unknown) definite, we may assume that at least one of a_0 , b_0 , c_0 and d_0 is not identically zero. Moreover, the functions a_n , b_n , c_n and d_n must satisfy the boundary conditions at $z = 0$ and $z = l$. Upon substituting the expansions (16) and (4), into the equations (3), we obtain two equations in both of which the terms in $x^{n+\sigma}$ and $x^{n+\sigma} \ln x$ must separately balance. From the lowest-order terms σ can be determined (cf. the indicial equation in the application of the Frobenius method to ODEs): the algebraic details may be found in Appendix A. We distinguish two cases. In the nonresonant case, by which we mean that the externally imposed driving frequency ω matches none of the eigenfrequencies ω_n of the $x = 0$ field line, only $\sigma = 0$ is permissible. In the case when $\omega = \omega_r$ for some r , which we shall call the resonant case, both $\sigma = 0$ and $\sigma = -1$ are possible. In both resonant and nonresonant cases, all the coefficients c_n and d_n are identically zero if $\sigma = 0$ and so there is no logarithmic term: hence this is the regular solution. The $\sigma = -1$ solution contains a logarithmic term and hence we call it the singular solution.

Nonresonant Case

To illustrate the way in which the Frobenius solution is developed, let us consider the nonresonant case. Then, as has already been stated and as is proved in Appendix A, $\sigma = 0$ and the c_n and d_n are identically zero. Thus equations (16) become

$$\xi_x = \sum_{n=0}^{\infty} a_n x^n, \quad \xi_y = \sum_{n=0}^{\infty} b_n x^n.$$

Substituting these expansions and the expansion (4) into equations (3) gives

$$\begin{aligned} & \sum_{n=0}^{\infty} (\mathcal{L}_0 a_n + x \mathcal{L}_1 a_n + x^2 \mathcal{L}_2 a_n + \dots) x^n \\ &= \frac{1}{i\lambda} \sum_{n=0}^{\infty} [n \mathcal{L}_0 b_n + (n+1) x \mathcal{L}_1 b_n \\ & \quad + (n+2) x^2 \mathcal{L}_2 b_n + \dots] x^{n-1} \end{aligned} \quad (17a)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} ((\mathcal{L}_0 - \lambda^2 B^2) b_n + x \mathcal{L}_1 b_n + x^2 \mathcal{L}_2 b_n + \dots) x^n \\ &= -i\lambda B^2 \sum_{n=0}^{\infty} n a_n x^{n-1}. \end{aligned} \quad (17b)$$

For the lowest-order terms (x^0) to balance in each equation requires that

$$\mathcal{L}_0 a_0 = \frac{1}{i\lambda} (\mathcal{L}_0 b_1 + \mathcal{L}_1 b_0), \quad (\mathcal{L}_0 - \lambda^2 B^2) b_0 = -i\lambda B^2 a_1; \quad (18)$$

for the x^1 terms to balance we must have

$$\begin{aligned} \mathcal{L}_0 a_1 + \mathcal{L}_1 a_0 &= \frac{2}{i\lambda} (\mathcal{L}_0 b_2 + \mathcal{L}_1 b_1 + \mathcal{L}_2 b_0), \\ (\mathcal{L}_0 - \lambda^2 B^2) b_1 + \mathcal{L}_1 b_0 &= -2i\lambda B^2 a_2; \end{aligned} \quad (19)$$

and so on. In this solution a_0 and b_0 are arbitrary functions (except that they must satisfy the boundary conditions). Since the governing equations are linear, we may consider the two solutions $a_0 \neq 0$, $b_0 = 0$ and $b_0 \neq 0$, $a_0 = 0$ separately. Recalling that in the nonresonant case, the inverse of $\mathcal{L}_0$ is well-defined (equation (12)), it is readily apparent that the pairs of equations (18), (19) (and so on) determine the functions a_n and b_n for $n = 1, 2, \dots$. Thus one deduces (see Appendix A for details) that the solution in the nonresonant case is a linear combination of the solution generated by a_0 and the solution generated by b_0 :

$$\begin{aligned} \xi_x &= a_0 - \frac{1}{i\lambda B^2} (\mathcal{L}_0 - \lambda^2 B^2) b_0 x \\ &+ \left[-\frac{1}{2B^2} (\mathcal{L}_0 - \lambda^2 B^2) a_0 + \frac{1}{2} i\lambda \mathcal{L}_0^{-1} \mathcal{L}_1 b_0 \right] x^2 \\ &+ O(x^3) \end{aligned} \quad (20a)$$

$$\begin{aligned} \xi_y &= b_0 + [i\lambda a_0 - \mathcal{L}_0^{-1} \mathcal{L}_1 b_0] x \\ &- \left[\frac{1}{2} i\lambda \mathcal{L}_0^{-1} \mathcal{L}_1 a_0 + \{ \mathcal{L}_0^{-1} \mathcal{L}_2 - (\mathcal{L}_0^{-1} \mathcal{L}_1)^2 \} b_0 \right. \\ & \quad \left. + \frac{1}{2B^2} (\mathcal{L}_0 - \lambda^2 B^2) b_0 \right] x^2 \\ &+ O(x^3), \end{aligned} \quad (20b)$$

where a_0 and b_0 are arbitrary functions of z . Furthermore, from equation (2b),

$$b_z = \frac{1}{i\lambda B} \mathcal{L}_0 b_0 + \frac{1}{B} \mathcal{L}_0 a_0 x + \frac{1}{2B} \left[\mathcal{L}_1 a_0 - \frac{1}{i\lambda B^2} (\mathcal{L}_0 - \lambda^2 B^2) \mathcal{L}_0 b_0 \right] x^2 + O(x^3). \quad (20c)$$

Resonant Case

Regular Solution. Henceforth we restrict attention to the resonant case ($\omega^2 = \omega_r^2$ for some r). Let us consider first the regular solution ($\sigma = 0$). Equations (19) and (20) still hold: the difference is that now $\mathcal{L}_0$ is invertible only if the solvability condition is satisfied. Once again a_0 is arbitrary; and b_0 is arbitrary except for a single restriction. The first of equations (18) can be rewritten as

$$\mathcal{L}_0(i\lambda a_0 - b_1) = \mathcal{L}_1 b_0; \quad (21)$$

and it follows from the solvability condition (13) that this is only solvable for b_1 if $\langle \phi_r \mathcal{L}_1 b_0 \rangle = 0$. Once again the solutions for $a_0 \neq 0$, $b_0 = 0$ and $b_0 \neq 0$, $a_0 = 0$ can be considered separately and then combined in any linear combination. For example, in the case $b_0 = 0$ it follows from equation (21) that

$$b_1 = i\lambda a_0 + \gamma \phi_r, \quad (22)$$

where the constant γ is (as yet) undetermined. The constant is determined by going to the next order: the solvability condition applied to the first of equations (19) gives

$$\langle \phi_r \mathcal{L}_1 b_1 \rangle = \frac{1}{2} i\lambda \langle \phi_r \mathcal{L}_1 a_0 \rangle \quad (23)$$

and hence, using equation (22),

$$\gamma = \frac{-i\lambda \langle \phi_r \mathcal{L}_1 a_0 \rangle}{2 \langle \phi_r \mathcal{L}_1 \phi_r \rangle}. \quad (24)$$

Continuing this procedure (see Appendix A for details), the regular solution in the resonant case is

$$\xi_x = a_0 - \frac{1}{i\lambda B^2} (\mathcal{L}_0 - \lambda^2 B^2) b_0 x + O(x^2) \quad (25a)$$

$$\xi_y = b_0 + [i\lambda a_0 + \beta_1 \phi_r - \mathcal{L}_0^{-1} \mathcal{L}_1 b_0 + \gamma_1 \phi_r] x + O(x^2), \quad (25b)$$

with

$$\beta_1 = -\frac{1}{2} i\lambda \langle \phi_r \mathcal{L}_1 a_0 \rangle / \langle \phi_r \mathcal{L}_1 \phi_r \rangle,$$

$$\gamma_1 = (\langle \phi_r \mathcal{L}_1 \mathcal{L}_0^{-1} \mathcal{L}_1 b_0 \rangle - \langle \phi_r \mathcal{L}_2 b_0 \rangle) / \langle \phi_r \mathcal{L}_1 \phi_r \rangle.$$

Also, from equation (2b),

$$b_z = \frac{1}{i\lambda B} \mathcal{L}_0 b_0 + \frac{1}{B} \mathcal{L}_0 a_0 x + O(x^2). \quad (25c)$$

The functions $a_0(z)$ and $b_0(z)$ are arbitrary except that $\langle \phi_r \mathcal{L}_1 b_0 \rangle$ must be zero.

Singular Solution. The singular solution ($\sigma = -1$) is a little more complicated algebraically because the coefficients c_n and d_n of the logarithmic terms are no longer identically zero. The details are given in Appendix A. Substituting the expansions (16) in equations (3) (cf. equations (A3)), and balancing terms of order $x^{\sigma-1}$, one obtains

$$\sigma a_0 + c_0 = 0, \quad \sigma \mathcal{L}_0 b_0 + \mathcal{L}_0 d_0 = 0;$$

and balancing the $x^{\sigma-1} \ln x$ terms yields

$$\sigma c_0 = 0, \quad \sigma \mathcal{L}_0 d_0 = 0.$$

Since $\sigma \neq 0$ for the singular solution, these equations imply that $a_0 = c_0 = 0$ and $\mathcal{L}_0 b_0 = \mathcal{L}_0 d_0 = 0$, i.e., b_0 and d_0 are multiples of the resonant eigenfunction ϕ_r . But at the next order one finds that

$$\mathcal{L}_0 d_1 + \mathcal{L}_1 d_0 = 0;$$

from which (using the solvability condition) one finds that $d_0 = 0$. Hence the only nonzero coefficient at lowest order is $b_0 = \beta_0 \phi_r$ (where β_0 is an arbitrary constant). From the remaining equations at this order, and proceeding to higher orders, one finds that

$$\xi_x = -i\lambda \beta_0 \phi_r \ln x + O(x, x^2 \ln x) \quad (26a)$$

$$\xi_y = \beta_0 \phi_r \frac{1}{x} - \beta_0 \frac{\langle \phi_r \mathcal{L}_2 \phi_r \rangle}{\langle \phi_r \mathcal{L}_1 \phi_r \rangle} \phi_r + \frac{1}{2} \lambda^2 \beta_0 \phi_r x \ln x + O(x, x^2 \ln x) \quad (26b)$$

$$b_z = \frac{\beta_0}{i\lambda B} \mathcal{L}_1 \phi_r + O(x^2 \ln x) \quad (26c)$$

The logarithmic nature of the singularity is in accord with the results of previous studies [Pao, 1975; Southwood, 1974; Kivelson and Southwood, 1986; Chen and Cowley, 1989; Mond et al., 1990; Wright, 1991].

It is possible to derive recursion formulae for the coefficients in the generalized Frobenius solutions. The details are given in Appendix B.

4. CONTINUING THE SOLUTION ACROSS THE SINGULARITY

To complete the singular solution it is necessary to determine how to continue the solution across the singularity. One way to do this is to allow the boundaries (at $z = 0, l$) to be weakly absorptive [Kivelson and Southwood, 1986; Mond et al., 1990]. In our approach, which is based upon that of Mond et al. [1990], we consider that the boundaries are not absorptive but rather that the system is weakly driven, so that the time dependence is $\exp(-i\omega t)$, where now $\omega = \omega_R + i\omega_I$ (ω_R real, $\omega_I > 0$). The growth time, which is of order $2\pi/\omega_I$, is assumed to be much greater than any Alfvén or fast mode transit time of the system. Here ω_R is the real driving frequency that was written in previous sections without a subscript. Thus ω_R^2 is equal to the square of the r th eigenfrequency of the $x = 0$ field line, viz., $\omega_R^2 = \omega_r^2(0)$.

When ω is complex, there is no longer an exact resonance at $x = 0$ (or any other real x). In a formal sense, the resonance now lies instead at a complex position $x_{\text{res}} = 0 + ix_i$. Of course, x_i is a function of ω_I . (Note that at this stage x_i may be complex.) We seek to determine the dependence of x_i upon ω_I , most importantly its sign.

In this section we shall write

$$B^2 \frac{\partial^2}{\partial z^2} + \mu_0 \rho_0(x, z) \omega^2 \equiv \mathcal{L}(\omega, x) \quad (27)$$

(cf. equation (1)); $\mathcal{L}$ has an implicit z dependence, which should be understood. As before, the operator $\mathcal{L}$ can be expanded in a Taylor expansion about the resonant position:

$$\mathcal{L}(\omega, x) = \mathcal{L}(\omega, x_{\text{res}}) + (x - x_{\text{res}}) \mathcal{L}_1(\omega, x_{\text{res}}) + \dots \quad (28)$$

Now $\mathcal{L}(\omega, x_{\text{res}})$ can in turn be expressed as a Taylor expansion about the original resonance:

$$\mathcal{L}(\omega, x_{\text{res}}) = \mathcal{L}(\omega_r, 0) + ix_i \mathcal{L}_1(\omega_r, 0) + i\omega_I \left. \frac{\partial \mathcal{L}}{\partial \omega} \right|_{(\omega_r, 0)} + \dots \quad (29)$$

The choice of driving frequency determines both the position x_{res} of the resonance and the form of the resonant eigenfunction ϕ_{res} , where

$$\mathcal{L}(\omega, x_{\text{res}})\phi_{\text{res}} = 0. \quad (30)$$

We can write ϕ_{res} in terms of the eigenfunction ϕ_r at $x = 0$ with eigenfrequency ω_r as

$$\phi_{\text{res}} = \phi_r + \Delta\phi_{\text{res}}. \quad (31)$$

Using expression (29) up to first order in small quantities for $\mathcal{L}(\omega, x_{\text{res}})$ and expression (31) for ϕ_{res} , equation (30) yields

$$ix_i \mathcal{L}_1(\omega_r, 0)\phi_r + i\omega_I \left. \frac{\partial \mathcal{L}}{\partial \omega} \right|_{(\omega_r, 0)} \phi_r + \mathcal{L}(\omega_r, 0)\Delta\phi_{\text{res}} = 0, \quad (32)$$

using $\mathcal{L}(\omega_r, 0)\phi_r = 0$. Multiplying by ϕ_r and integrating with respect to z at $x = 0$, we find

$$x_i = -\omega_I \frac{\langle \phi_r \partial \mathcal{L} / \partial \omega |_{(\omega_r, 0)} \phi_r \rangle}{\langle \phi_r \mathcal{L}_1(\omega_r, 0) \phi_r \rangle}, \quad (33)$$

where the operators $\partial \mathcal{L} / \partial \omega$ and $\mathcal{L}_1$ are evaluated at $\omega = \omega_r$ and $x = 0$. Note that the final step is possible because $\Delta\phi_{\text{res}}$ satisfies the same perfectly reflecting boundary conditions as ϕ_r at $z = 0, l$. If the boundaries were weakly absorptive, there would be an additional term in equation (33). For our differential operator, equation (33) may be rewritten as

$$x_i = -\frac{2\omega_I}{\omega_r} \frac{\langle \phi_r \rho |_{x=0} \phi_r \rangle}{\langle \phi_r \partial \rho / \partial x |_{x=0} \phi_r \rangle}; \quad (34)$$

here ρ and $\partial \rho / \partial x$ are evaluated at $x = 0$. Note that equation (34) implies that x_i is real to this order.

The Frobenius series solutions in the weakly driven case are the same as before, except that instead of being expansions of powers of $(x - 0)$ and its logarithm, they are expansions in powers of $(x - x_{\text{res}}) \equiv (x - ix_i)$ and its logarithm. Consider then the argument of $\ln(x - ix_i)$ as one goes from $\text{Re}[x] < 0$ to $\text{Re}[x] > 0$. If x_i is positive, $\arg(x - ix_i)$ increases; whereas if x_i is negative it decreases. Finally we let $\omega_I \rightarrow 0$: $\arg(x - ix_i)$ increases/decreases by π on moving from $x = 0^-$ to $x = 0^+$, and this determines how to continue the solution across the resonance in the case of a real driving frequency.

Having taken the limit $\omega_I \rightarrow 0$, the x component of the plasma displacement is found to be $\xi_x \simeq -i\lambda\beta_0\phi_r \ln|x|$ at $x = 0^-$ and $\xi_x \simeq -i\lambda\beta_0\phi_r(\ln|x| + i\pi\text{sign}[x_i])$ at $x = 0^+$. (Of course, we mean $\text{sign}[x_i]$, $\text{sign}[\omega_I]$ to be evaluated before taking the limit.) Now the x component of the Poynting flux, S_x , which will not time average to zero is $\mu_0^{-1} B \text{Re}[b_z \exp i(\lambda y - \omega t)] \text{Re}[v_x \exp i(\lambda y - \omega t)]$, where v_x is the plasma velocity in the x direction. After averaging with respect to time this is equivalent to $iB\omega(\xi_x^* b_z - \xi_x b_z^*) / (4\mu_0)$ where the perturbations in this expression have no implicit time dependence; i.e., they are the functions given in (20) and (25). An effect of the change in phase across the resonance is that the Poynting flux is discontinuous there: the difference between S_x (integrated along the $x = 0$ field line) on the two sides of $x = 0$ is

$$\langle S_x(0^-) \rangle - \langle S_x(0^+) \rangle = \frac{-\pi}{2\mu_0} \omega |\beta_0|^2 \langle \phi_r \mathcal{L}_1 \phi_r \rangle \text{sign}[x_i], \quad (35)$$

by equations (26). At the end of this section we show that

the difference in the above energy fluxes is positive, so the resonance absorbs energy. (The regular solution gives a continuous contribution to the Poynting flux and so does not affect this calculation.)

In the previous studies of ODE resonance problems [Southwood, 1974], it is clear that the change in phase of the argument of the logarithm depends solely upon the gradient of the r th Alfvén eigenfrequency at the resonance. The choice of sign of the phase change means that the resonance absorbs energy, rather than radiates energy. It is not evident from our more general analysis (see (33) and (34)) how x_i depends on $d\omega_r^2(x)/dx$. For example, is it possible to specify an $\mathcal{L}$ which will yield an arbitrary eigenfrequency gradient across the resonance? The factor $\langle \phi_r \mathcal{L}_1 \phi_r \rangle$ enters many of our expressions (e.g., equation (33)), and it is worth developing an interpretation for it. Let the r th eigenmode and eigenfrequency satisfy equation (6) at $x = 0$. Now consider the change in eigenmode and eigenfrequency on an adjacent field line at $x = \delta x$ (δx real). The change to the density, eigenfrequency and eigenmode relative to those on $x = 0$ are $\delta\rho_0 = \delta x(\partial\rho_0(x, z)/\partial x)_0$, $\delta\omega_r^2 = \delta x(d\omega_r^2/dx)_0$ and $\delta\phi_r = \delta x(\partial\phi_r(x, z)/\partial x)_0$ respectively, all the derivatives being evaluated at $x = 0$. Substituting these changes in to a Taylor series expansion of $\mathcal{L}\phi_r = 0$ about $x = 0$ we find the terms linear in δx satisfy

$$\mathcal{L}_0 \left. \frac{\partial \phi_r}{\partial x} \right|_{x=0} + \left[\mathcal{L}_1 + \mu_0 \rho_0(0, z) \left. \frac{d\omega_r^2}{dx} \right|_{x=0} \right] \phi_r(0, z) = 0. \quad (36)$$

Multiplying equation (36) by ϕ_r and integrating along the background field line $x = 0$ we find the simple relation

$$\left. \frac{d\omega_r^2}{dx} \right|_{x=0} = -\frac{\langle \phi_r \mathcal{L}_1 \phi_r \rangle}{\mu_0 \langle \phi_r \rho_0 \phi_r \rangle}. \quad (37)$$

Recalling the definition of $\mathcal{L}_1$ it is clear from (34) and (36) that $x_i = \omega_I / (d\omega_r/dx)$. Thus, as one would anticipate, the sense of the phase change of the log terms across the resonance is determined by the gradient of the resonant eigenfrequency.

Note that the signs of x_i , ω_I and $d\omega_r/dx$ are related by $\text{sign}[d\omega_r/dx] = \text{sign}[x_i]$, since ω_I is by assumption positive. Employing this relation (35) and (37) may be combined to yield an expression for the power absorbed over the length of the resonant field line, per unit length in the y direction

$$\langle S_x(0^-) \rangle - \langle S_x(0^+) \rangle = \pi\omega^2 |\beta_0|^2 \langle \phi_r \rho_0 \phi_r \rangle \left. \frac{d\omega_r}{dx} \right|_0 \quad (38)$$

The above energy flux is positive definite: thus the resonance absorbs energy, which is in accord with the original one-dimensional calculations [Chen and Hasegawa, 1974b; Southwood, 1974].

5. DISCUSSION

It is instructive to compare our results with those of previous workers, who have in the main concentrated on the singular solution. Southwood and Kivelson [1986] and Wright [1992a,b] have viewed the magnetic field perturbation b_z as driving the system given by equations (2a,b). From our solutions in (25c) and (26c) it is evident that the leading-order of b_z is $O(x^0)$, which we shall denote as $b_{x0}(z)$. We can relate the leading-order singular solution for ξ_y , namely $\beta_0\phi_r/x$, to b_{x0} by considering the expanded form of (2b). Multiplying by ϕ_r and integrating in z we find

$$\beta_0 = i\lambda B \frac{\langle \phi_r b_{z0} \rangle}{\langle \phi_r \mathcal{L}_1 \phi_r \rangle} = \frac{-i\lambda B \langle \phi_r b_{z0} \rangle}{2\mu_0 \omega_r \langle \phi_r \rho_0 \phi_r \rangle} \left(\frac{d\omega_r}{dx} \right)_0^{-1} \quad (39)$$

(The second equality uses equation [37].) Note how the parameter β_0 depends upon the overlap integral of b_{z0} and the resonant eigenfunction; β_0 represents the amplitude of the singular (or resonant) response, and is evidently sensitive to the variation of b_{z0} along the resonant field line. This property affirms previous results [Southwood and Kivelson, 1986; Chen and Cowley, 1989; Wright, 1991]. It is instructive to note here that while the singular solution dominates the displacement ξ near the resonance and has the z dependence of the resonant eigenfunction, there is no such restriction on the z dependence of b_z : since both the singular and regular solutions for b_z start at order x^0 , b_z may have any z dependence on the resonant field line. (Note though that the part of b_z which comes from the regular solution makes no contribution to β_0 , as may be verified by substituting the leading-order regular solution (25c) into the right-hand side of equation (39).) Thus if, for example, the resonant eigenfunction is an even function about the centre of the field line, but b_{z0} is an odd function, then there will be no resonant excitation of Alfvén waves since $\langle \phi_r b_{z0} \rangle = 0$. In general the factor $\langle \phi_r b_{z0} \rangle$ represents the efficiency of coupling between the b_z (i.e., the fast mode) and the Alfvén resonance [Southwood and Kivelson, 1986; Wright, 1992a]. Indeed Southwood and Kivelson [1986] suggested that in some density distributions the fast mode (with a fundamental field aligned dependence) may couple most efficiently to the third harmonic Alfvén mode. (Note that in the case where the density is independent of z , such a fast mode could only couple to the fundamental Alfvén mode.)

Now that we have derived an expression for β_0 we may write a final expression for the Poynting flux absorbed by the resonant layer. Substituting for β_0 in equation (38) gives

$$\langle S_x(0^-) \rangle - \langle S_x(0^+) \rangle = \frac{\pi \lambda^2 B^2}{4\mu_0^2} \frac{|\langle \phi_r b_{z0} \rangle|^2}{\langle \phi_r \rho_0 \phi_r \rangle} \left| \frac{d\omega_r}{dx} \right|_{x=0}^{-1} \quad (40)$$

The expression above demonstrates many features found in previous studies, such as the dependence on λ^2 , the square of the overlap integral representing the efficiency of coupling, and the inverse dependence upon $d\omega_r/dx$ (see, for example, equation (18) of Wright [1992a]).

Some previous studies have been able to deduce the leading behavior of ξ_x and ξ_y for the singular solution. Although our method enables us to determine the behavior at all orders for these displacements, we list the leading order terms here to facilitate comparison with existing results. From equations (26) and (39) we see

$$\xi_x \approx \frac{-\lambda^2 B \langle \phi_r b_{z0} \rangle}{2\mu_0 \omega_r \langle \phi_r \rho_0 \phi_r \rangle} \left(\frac{d\omega_r}{dx} \right)_0^{-1} \phi_r(0, z) \ln(x) \quad (41a)$$

$$\xi_y \approx \frac{-i\lambda B \langle \phi_r b_{z0} \rangle}{2\mu_0 \omega_r \langle \phi_r \rho_0 \phi_r \rangle} \left(\frac{d\omega_r}{dx} \right)_0^{-1} \frac{\phi_r}{x} \quad (41b)$$

$$b_z \approx b_{z0} \quad (41c)$$

Note that these expressions are for quite arbitrary density, $\rho_0(x, z)$.

Southwood and Kivelson [1986] produced a similar expression to our (41b) when they imposed the simplification of having a density distribution that was a separable function of x and z . To compare with their results we shall write $\rho_0(x, z) = \rho_x(x)\rho_z(z)$. From the structure of the eigenvalue

equation it follows that the n th eigenfrequency (which will be a function of x) will be determined by the variation of $\rho_x(x)$; The quantity $\omega_n^2 \rho_x$ is the pseudo-eigenvalue for the operator $B^2 \partial^2 / \partial z^2 + [\omega_n^2 \rho_x] \mu_0 \rho_x(z)$ and is independent of x . Thus, from

$$\omega_n^2(x) \rho_x(x) = \text{const} \quad (42)$$

we deduce that

$$\frac{d\omega_n^2}{dx} = -\frac{\omega_n^2}{\rho_x} \frac{d\rho_x}{dx}. \quad (43)$$

For the resonant eigenmode we shall require $\omega^2 = \omega_r^2(0)$. Recalling the definition of the operator $\mathcal{L}_1$ (equation [5]) and employing the relation (43), $\mathcal{L}_1$ may be expressed in the form

$$\mathcal{L}_1 = -\mu_0 \rho_x(z) \rho_x(0) \left[\frac{d\omega_r^2}{dx} \right]_{x=0}. \quad (44)$$

By defining $[d\omega_r^2/dx]_0 = (\omega_r^2(x) - \omega^2)/x$ in the limit $x \rightarrow 0$, we find

$$\mathcal{L}_1 = -\mu_0 \rho_x(z) \rho_x(0) \frac{\omega_r^2(x) - \omega^2}{x}; \quad (45)$$

and equation (41b) gives the leading behavior of ξ_y as

$$\xi_y \approx \frac{i\lambda B \langle \phi_r b_{z0} \rangle}{\mu_0 \rho_x(0) \langle \phi_r \rho_x(z) \phi_r \rangle} \frac{\phi_r}{[\omega^2 - \omega_r^2(x)]}. \quad (46)$$

The above expression is identical to equation (34) of Southwood and Kivelson [1986], if we assume that their normalised eigenfunctions satisfy $\langle \phi_r \rho_x(z) \phi_r \rangle = 1$. Thus we are able to recover the existing results of Southwood and Kivelson [1986] by taking the appropriate limit of our more general analysis.

Note that the amplitudes of the singular solution fields in equations (41a) and (41b) are inversely proportional to $|d\omega_r/dx|$, suggesting that our singular solution analysis breaks down when

$$\left. \frac{d\omega_r}{dx} \right|_{x=0} = 0. \quad (47)$$

This is indeed the case. When the above relation is satisfied we conclude that not only is the field line at $x = 0$ resonant, but field lines close to $x = 0$ appear, at lowest order, to be resonant too. It should be noted that the proof of $\sigma = -1$ for the singular solution (Appendix A) relied upon the property $\langle \phi_r \mathcal{L}_1 \phi_r \rangle \neq 0$ when we invoked the solvability condition. When considering the situation in which (47) is valid, however, equation (37) implies that $\langle \phi_r \mathcal{L}_1 \phi_r \rangle = 0$ and hence we cannot assume that $\sigma = -1$ for the singular solution. The method set down in Appendix A might be extended to this case. However, it is instructive instead to deduce the new singular index value in the following manner, provided we assume that the lowest-order behavior of b_z is still of order 1 (say, $b_{z0}(z)$, again). Expanding $\mathcal{L}$ according to (5), and integrating the product of ϕ_r with equation (2b) along the field line at $x = 0$ we find balancing powers of x will arise from the following terms

$$\langle \phi_r (\mathcal{L}_0 + x\mathcal{L}_1 + x^2\mathcal{L}_2 + \dots) (\beta_0 \phi_r x^\sigma + b_1 x^{\sigma+1} + b_2 x^{\sigma+2} + \dots) \rangle = i\lambda B \langle b_{z0}(z) \phi_r \rangle, \quad (48)$$

where we have used the property that $b_0 = \beta_0 \phi_r(0, z)$ if $\sigma \neq 0$ (see (A5)). Evidently, when $\langle \phi_r \mathcal{L}_1 \phi_r \rangle \neq 0$, the term that is $O(1)$ on the right-hand side of equation (48) is balanced by a term on the left-hand side which requires $\sigma = -1$. However, when $\langle \phi_r \mathcal{L}_1 \phi_r \rangle = 0$, balancing terms on

both sides requires that $\sigma = -2$; but then only provided that $\langle \phi_r [\mathcal{L}_1 b_1 + \mathcal{L}_2 b_0] \phi_r \rangle$ is nonzero. If this factor were zero, we would have to try and balance terms with $\sigma = -3$.

In the case where equation (47) holds and the singular solution has an index of -2 , we find the leading order behavior of ξ_y to be

$$\xi_y \approx \beta_0 \frac{\phi_r}{x^2}. \quad (49)$$

while from equation (2c) the behavior of ξ_x is

$$\xi_x \approx i\lambda\beta_0 \frac{\phi_r}{x}. \quad (50)$$

For example, such a circumstance would arise in the box model [Southwood, 1974] if $\partial\rho/\partial x$ vanished at the resonant field line (provided $\partial^2\rho/\partial x^2$ did not also vanish). In that case, the solutions for ξ_x , ξ_y and b_z contain no logarithmic terms and hence no Poynting flux is absorbed by the layer. It is easy to investigate the character of the singular solution of the one-dimensional box model (governed by an ODE). More generally, we find that if the first nonzero derivative of the density is the N th, then there are no logarithmic terms in the singular solution when N is even, and no energy absorption at the resonance ($\xi_x \sim 1/x^{N-1}$; $\xi_y \sim 1/x^N$). When N is odd there is a logarithmic term in the singular solution and a net absorption of energy by the resonant field lines ($\xi_x \sim 1/x^{N-1} \dots + \ln(x)$; $\xi_y \sim 1/x^N$). It would appear that energy absorption at a resonance is dependent upon having a monotonic change in natural Alfvén frequency across the resonance.

6. CONCLUSIONS

We have presented a systematic method for determining the regular and singular solutions at the singularity in a partial differential equation describing the resonant excitation of Alfvén waves. Future work will generalize this analysis and enable us to consider more general magnetoplasmas (e.g., curl-free magnetic fields, force-free magnetic fields, or warm plasmas).

Previous authors [Southwood and Kivelson, 1986; Chen and Cowley, 1989; Wright, 1991; Wright, 1992a] have demonstrated the importance of the “overlap integral” $\langle b_{z0} \phi_r \rangle$ which describes how efficiently the fast mode may drive a resonance. This quantity appears quite naturally in our series solution. It may be noted that our derivation does not impose any of the simplifications of earlier studies: namely that ρ_0 is a separable function of x and z , or that the wave number λ is small.

Note added in proof. During publication of this paper an article by Hansen and Goertz (Phys. Fluids 4, 2713, 1992) appeared which called into question the validity of the “field line resonance” (FLR) expansion. This refers to the method of solution employed by, for example, Chen and Cowley [1989] and Mond et al. [1990]. Hansen and Goertz argue that the FLR expansion leads to an inconsistent solution. Hansen and Goertz substantiate their claim by using a similar generalised Frobenius series to ours (their equation (31); we note, however, that in that equation they omit terms – presumably because they are regular – which nonetheless should be included because they would contribute in later equations at the same order as terms which they have considered). Our present paper solves explicitly for the coefficients

in this series, which Hansen and Goertz did not do. We demonstrate the existence of a singular solution with leading order variation along the resonant field line of the resonant eigenfunction, in contradiction to the conclusions of Hansen and Goertz.

APPENDIX A: GENERALIZED FROBENIUS SOLUTIONS FOR THE REGULAR AND SINGULAR SOLUTIONS

For completeness we restate the governing coupled equations that we wish to solve:

$$\frac{\partial \xi_x}{\partial x} = \frac{-1}{i\lambda B^2} (\mathcal{L}\xi_y - \lambda^2 B^2 \xi_y) \quad (A1a)$$

$$\frac{\partial \mathcal{L}\xi_y}{\partial x} = i\lambda \mathcal{L}\xi_x \quad (A1b)$$

(cf. equations (3)). We assume that the solution can be written as an expansion of the form

$$\begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} = x^\sigma \sum_{n=0}^{\infty} \begin{pmatrix} a_n(z) \\ b_n(z) \end{pmatrix} x^n + x^\sigma \ln x \sum_{n=0}^{\infty} \begin{pmatrix} c_n(z) \\ d_n(z) \end{pmatrix} x^n \quad (A2)$$

(cf. equations (16)).

For definiteness in determining σ , at least one of a_0 , b_0 , c_0 and d_0 must be nonzero. Substituting the expansion (A2) into equations (A1) yields

$$\begin{aligned} & -i\lambda B^2 [(\sigma a_0 + c_0)x^{\sigma-1} + ((\sigma+1)a_1 + c_1)x^\sigma \\ & + ((\sigma+2)a_2 + c_2)x^{\sigma+1} + \dots] - i\lambda B^2 \ln x [\sigma c_0 x^{\sigma-1} \\ & + (\sigma+1)c_1 x^\sigma + (\sigma+2)c_2 x^{\sigma+1} + \dots] \\ & = (\mathcal{L}_0 - \lambda^2 B^2)b_0 x^\sigma + \{(\mathcal{L}_0 - \lambda^2 B^2)b_1 + \mathcal{L}_1 b_0\} x^{\sigma+1} \\ & + \dots + \ln x [(\mathcal{L}_0 - \lambda^2 B^2)d_0 x^\sigma \\ & + \{(\mathcal{L}_0 - \lambda^2 B^2)d_1 + \mathcal{L}_1 d_0\} x^{\sigma+1} + \dots] \end{aligned} \quad (A3a)$$

$$\begin{aligned} & [(\sigma \mathcal{L}_0 b_0 + \mathcal{L}_0 d_0)x^{\sigma-1} + \{(\sigma+1)(\mathcal{L}_0 b_1 + \mathcal{L}_1 b_0) \\ & + (\mathcal{L}_0 d_1 + \mathcal{L}_1 d_0)\} x^\sigma + \{(\sigma+2)(\mathcal{L}_0 b_2 + \mathcal{L}_1 b_1 + \mathcal{L}_2 b_0) \\ & + (\mathcal{L}_0 d_2 + \mathcal{L}_1 d_1 + \mathcal{L}_2 d_0)\} x^{\sigma+1} + \dots] \\ & + \ln x [(\sigma \mathcal{L}_0 d_0)x^{\sigma-1} + (\sigma+1)(\mathcal{L}_0 d_1 + \mathcal{L}_1 d_0)x^\sigma \\ & + (\sigma+2)(\mathcal{L}_0 d_2 + \mathcal{L}_1 d_1 + \mathcal{L}_2 d_0)x^{\sigma+1} + \dots] \\ & = i\lambda [(\mathcal{L}_0 a_0)x^\sigma + (\mathcal{L}_0 a_1 + \mathcal{L}_1 a_0)x^{\sigma+1} + \dots] \\ & + i\lambda \ln x [(\mathcal{L}_0 c_0)x^\sigma + (\mathcal{L}_0 c_1 + \mathcal{L}_1 c_0)x^{\sigma+1} + \dots] \end{aligned} \quad (A3b)$$

In the nonresonant case, $\mathcal{L}_0 f = 0$ implies that $f = 0$, and $\mathcal{L}_0 f = g$ can always be inverted to infer f in terms of g . In the resonant case, on the other hand, there exists a resonant eigenfunction ϕ_r such that $\mathcal{L}_0 \phi_r = 0$. In this case, the equation $\mathcal{L}_0 f = g$ is not always solvable for f , as has already been noted in section 2: in fact it is only solvable if $\langle \phi_r g \rangle = 0$, in which circumstance we adopt the definitions (14) and (15) for the inverse operator.

Throughout this paper we have to invert equations of the form $\mathcal{L}_0 f + \mathcal{L}_1 \gamma \phi_r = 0$. The solvability condition, equation (13), requires that

$$\gamma \langle \phi_r \mathcal{L}_1 \phi_r \rangle = 0. \quad (A4)$$

At the end of section 4 it is shown that $\langle \phi_r \mathcal{L}_1 \phi_r \rangle \neq 0$ is equivalent to stating $[d\omega_r^2(x)/dx]_0 \neq 0$, which we would expect to be true for most media. The consequences of $[d\omega_r^2(x)/dx]_0 = \langle \phi_r \mathcal{L}_1 \phi_r \rangle = 0$ are discussed further in sec-

tion 5. For the principal calculation in this paper we shall assume that $\langle \phi_r \mathcal{L}_1 \phi_r \rangle \neq 0$: hence equation (A4) implies that $\gamma = 0$.

Allowable values of the index σ

Balancing $x^{\sigma-1}$ terms and $x^{\sigma-1} \ln x$ terms in equations (A3) requires that

$$\sigma a_0 = 0, \quad \sigma \mathcal{L}_0 b_0 = 0, \quad c_0 = 0 \quad \text{and} \quad \mathcal{L}_0 d_0 = 0. \quad (\text{A5})$$

Nonresonant case. In this case, $\mathcal{L}_0$ is invertible. Thus, if $\sigma \neq 0$, equations (A5) imply that $a_0 = b_0 = c_0 = d_0 = 0$, which contradicts the assumption that these functions are not all identically zero. Hence in the nonresonant case σ must be zero.

Resonant case. Suppose that $\sigma \neq 0$ and $\sigma \neq -1$. Equations (A5) then imply that $a_0 = c_0 = 0$ and $\mathcal{L}_0 b_0 = \mathcal{L}_0 d_0 = 0$. Thus b_0 and d_0 are proportional to ϕ_r . Balancing the x^σ terms and $x^\sigma \ln x$ in equations (A3) then gives

$$(\sigma + 1)(\mathcal{L}_0 b_1 + \mathcal{L}_1 b_0) + (\mathcal{L}_0 d_1 + \mathcal{L}_1 d_0) = 0 \quad (\text{A6a})$$

$$(\sigma + 1)(\mathcal{L}_0 d_1 + \mathcal{L}_1 d_0) = 0, \quad (\text{A6b})$$

which (since we are supposing that $\sigma \neq -1$) implies

$$(\mathcal{L}_0 b_1 + \mathcal{L}_1 b_0) = 0 \quad (\text{A7a})$$

$$(\mathcal{L}_0 d_1 + \mathcal{L}_1 d_0) = 0. \quad (\text{A7b})$$

The solvability condition (SC) then implies that $b_0 = d_0 = 0$, which contradicts the assumption that at least one of a_0 , b_0 , c_0 and d_0 must be nonzero. Therefore, in the resonant case, σ must take the value 0 or -1 .

As will become evident, $\sigma = 0$ and $\sigma = -1$ provide the regular and singular solutions respectively.

Absence of Logarithmic Terms in the Regular ($\sigma = 0$) Solution

Consider $\sigma = 0$. Balancing the $x^{n-1} \ln x$ terms in equations (A3) yields

$$i\lambda B^2 n c_n = -(\mathcal{L}_0 d_{n-1} + \dots + \mathcal{L}_{n-1} d_0) + \lambda^2 B^2 d_{n-1} \quad (\text{A8a})$$

$$n \mathcal{L}_0 d_n = -n(\mathcal{L}_1 d_{n-1} + \dots + \mathcal{L}_n d_0) + i\lambda(\mathcal{L}_0 c_{n-1} + \dots + \mathcal{L}_{n-1} c_0) \quad (\text{A8b})$$

for $n \geq 1$. Moreover, from equations (A5) $c_0 = \mathcal{L}_0 d_0 = 0$. We now prove by induction that the functions c_n , d_n are all identically zero.

Proof. Suppose that $c_n = 0$ for all $n \leq N$, that $d_n = 0$ for all $n \leq N-1$ and that $\mathcal{L}_0 d_N = 0$. (This is certainly the case for $N = 0$.) Then we shall show that $c_{N+1} = 0$, that $d_N = 0$ and that $\mathcal{L}_0 d_{N+1} = 0$. Setting $n = N+1$ in equation (A8a) yields $c_{N+1} = 0$. In the nonresonant case $d_N = 0$ follows trivially from $\mathcal{L}_0 d_N = 0$; and setting $n = N+1$ in equation (A8b) yields $\mathcal{L}_0 d_{N+1} = 0$. In the resonant case, setting $n = N+1$ in equation (A8b) gives, for the regular solution ($\sigma = 0$),

$$\mathcal{L}_0 d_{N+1} + \mathcal{L}_1 d_N = 0;$$

hence from the solvability condition $d_N = 0$ and $\mathcal{L}_0 d_{N+1} =$

0, as required. Thus by mathematical induction all the c_n and d_n are zero, if $\sigma = 0$. QED

This justifies our use of the term "regular" to describe $\sigma = 0$ solutions.

Solutions in Nonresonant Case: Lowest Order Terms

When $\sigma = 0$, (so that $c_0 = d_0 = 0$) balancing the x^0 terms in equations (A3) yields

$$-i\lambda B^2 a_1 = (\mathcal{L}_0 - \lambda^2 B^2) b_0 \quad (\text{A9a})$$

$$\mathcal{L}_0(b_1 - i\lambda a_0) = -\mathcal{L}_1 b_0, \quad (\text{A9b})$$

and balancing the x^1 terms yields

$$\begin{aligned} -2i\lambda B^2 a_2 &= (\mathcal{L}_0 - \lambda^2 B^2) b_1 + \mathcal{L}_1 b_0 \\ &= -\lambda^2 B^2 b_1 + i\lambda \mathcal{L}_0 a_0 \end{aligned} \quad (\text{A9c})$$

$$\mathcal{L}_0(2b_2 - i\lambda a_1) = -2(\mathcal{L}_1 b_1 + \mathcal{L}_2 b_0) + i\lambda \mathcal{L}_1 a_0. \quad (\text{A9d})$$

In the nonresonant case, a_0 and b_0 are arbitrary functions of z . Since the governing equations are linear, we may consider the two solutions $a_0 \neq 0$, $b_0 = 0$ and $b_0 \neq 0$, $a_0 = 0$ separately: the general solution is a linear combination of these two.

Solution with $b_0 = 0$. Since $\mathcal{L}_0$ is invertible in the nonresonant case, equations (A9a) and (A9b) give

$$a_1 = 0, \quad b_1 = i\lambda a_0,$$

and equations (A9c) and (A9d) give

$$a_2 = \frac{-1}{2B^2}(\mathcal{L}_0 - \lambda^2 B^2)a_0, \quad b_2 = \frac{-i\lambda}{2}\mathcal{L}_0^{-1}(\mathcal{L}_1 a_0).$$

Solution with $a_0 = 0$. Equations (A9a) and (A9b) give

$$a_1 = \frac{-1}{i\lambda B^2}(\mathcal{L}_0 - \lambda^2 B^2)b_0, \quad b_1 = -\mathcal{L}_0^{-1}(\mathcal{L}_1 b_0),$$

and it follows from equations (A9c) and (A9d) that

$$a_2 = \frac{1}{2}i\lambda \mathcal{L}_0^{-1}(\mathcal{L}_1 b_0),$$

$$b_2 = \mathcal{L}_0^{-1}(\mathcal{L}_1 \mathcal{L}_0^{-1} \mathcal{L}_1 b_0 - \mathcal{L}_2 b_0) - \frac{1}{2B^2}(\mathcal{L}_0 - \lambda^2 B^2)b_0.$$

Regular ($\sigma = 0$) Solutions in Resonant Case: Lowest Order Terms

In this case the equations (A9a)-(A9d) still apply. We supplement these with two further equations, obtained by balancing x^2 terms in equations (A3):

$$\begin{aligned} -3i\lambda B^2 a_3 &= (\mathcal{L}_0 - \lambda^2 B^2) b_2 + \mathcal{L}_1 b_1 + \mathcal{L}_2 b_0 \\ &= -\lambda^2 B^2 b_2 + \frac{1}{2}i\lambda(\mathcal{L}_0 a_1 + \mathcal{L}_1 a_0) \end{aligned} \quad (\text{A9e})$$

$$\begin{aligned} \mathcal{L}_0(3b_3 - i\lambda a_2) &= -3(\mathcal{L}_1 b_2 + \mathcal{L}_2 b_1 + \mathcal{L}_3 b_0) \\ &\quad + i\lambda(\mathcal{L}_1 a_1 + \mathcal{L}_2 a_0). \end{aligned} \quad (\text{A9f})$$

Once again the function a_0 is arbitrary. From (A9b) and the solvability condition, b_0 must satisfy $\langle \phi_r \mathcal{L}_1 b_0 \rangle = 0$; but otherwise it is arbitrary. As in the nonresonant case, we can consider the cases $a_0 = 0$ and $b_0 = 0$ separately. By linearity, the general regular solution is a linear combination of these two solutions.

Solution with $b_0 = 0$. It follows from equations (A9a) and (A9b) that

$$a_1 = 0, \quad b_1 = i\lambda a_0 + \beta_1 \phi_r,$$

where β_1 is a constant that has yet to be determined. From equation (A9d) and the solvability condition we deduce that

$$\begin{aligned} 0 &= \langle \phi_r \mathcal{L}_1 (2b_1 - i\lambda a_0) \rangle \\ &= 2\beta_1 \langle \phi_r \mathcal{L}_1 \phi_r \rangle + i\lambda \langle \phi_r \mathcal{L}_1 a_0 \rangle, \end{aligned}$$

whence

$$\beta_1 = \frac{-i\lambda \langle \phi_r \mathcal{L}_1 a_0 \rangle}{2\langle \phi_r \mathcal{L}_1 \phi_r \rangle}.$$

Equation (A9c) gives

$$a_2 = \frac{-1}{2B^2} (\mathcal{L}_0 - \lambda^2 B^2) a_0 - \frac{1}{2} i\lambda \beta_1 \phi_r,$$

and equation (A9d) yields

$$b_2 = \frac{-1}{2} \mathcal{L}_0^{-1} (\mathcal{L}_1 (i\lambda a_0 + 2\beta_1 \phi_r)) + \beta_2 \phi_r.$$

Note that β_1 was defined in a manner such that the inverse $\mathcal{L}_0^{-1}$ is well-defined in this equation. (The inverse operator is defined by equations (14) and (15).) The constant β_2 follows from equation (A9f) and the solvability condition:

$$\begin{aligned} \beta_2 &= \left\{ -2\langle \phi_r \mathcal{L}_2 (2i\lambda a_0 + 3\beta_1 \phi_r) \rangle \right. \\ &\quad \left. + 3\langle \phi_r \mathcal{L}_1 \mathcal{L}_0^{-1} \mathcal{L}_1 (i\lambda a_0 + 2\beta_1 \phi_r) \rangle \right\} / 6\langle \phi_r \mathcal{L}_1 \phi_r \rangle \end{aligned}$$

Solution with $a_0 = 0$. We recall that b_0 is arbitrary except for the single restriction

$$\langle \phi_r \mathcal{L}_1 b_0 \rangle = 0.$$

Equations (A9a) and (A9b) imply that

$$a_1 = \frac{-1}{i\lambda B^2} (\mathcal{L}_0 - \lambda^2 B^2) b_0, \quad b_1 = -\mathcal{L}_0^{-1} (\mathcal{L}_1 b_0) + \gamma_1 \phi_r,$$

where the constant γ_1 is determined from applying the solvability condition to equation (A9d):

$$\gamma_1 = \frac{\langle \phi_r \mathcal{L}_1 \mathcal{L}_0^{-1} \mathcal{L}_1 b_0 \rangle - \langle \phi_r \mathcal{L}_2 b_0 \rangle}{\langle \phi_r \mathcal{L}_1 \phi_r \rangle}.$$

Singular ($\sigma = -1$) Solution: Lowest Order Terms

In the case $\sigma = -1$, equations (A5) give

$$a_0 = 0, \quad b_0 = \beta_0 \phi_r, \quad c_0 = 0, \quad d_0 = \delta_0 \phi_r, \quad (\text{A10})$$

where β_0 and δ_0 are constants. Equation (A6a) still holds; and hence by the solvability condition (noting that $\sigma + 1 = 0$ and once again assuming $\langle \phi_r \mathcal{L}_1 \phi_r \rangle \neq 0$), $\delta_0 = 0$ and so $d_0 = 0$. (One can also get this from the $x^\sigma \ln x$ term in equation [A3a], without invoking the SC.) The constant β_0 is undetermined.

The only remaining nontrivial equations that come from balancing x^σ terms and $x^\sigma \ln x$ terms in equations (A3) are

$$-i\lambda B^2 c_1 = -\lambda^2 B^2 b_0, \quad \mathcal{L}_0 d_1 = 0; \quad (\text{A11})$$

whence

$$c_1 = -i\lambda \beta_0 \phi_r, \quad d_1 = \delta_1 \phi_r \quad (\text{A12})$$

(with δ_1 still to be determined). The function a_1 is undetermined. But a_1 in the singular solution is the coefficient of x^0 in ξ_x , as is a_0 in the regular solution. Since a_0 in the regular solution is quite arbitrary, we may without loss of generality set a_1 here to be zero: this corresponds to adding a chosen amount of regular solution to the singular solution.

Going to the next order we obtain

$$-i\lambda B^2 (a_2 + c_2) = (\mathcal{L}_0 - \lambda^2 B^2) b_1 + \beta_0 \mathcal{L}_1 \phi_r \quad (\text{A13a})$$

$$-i\lambda B^2 c_2 = -\lambda^2 B^2 (\delta_1 \phi_r) \quad (\text{A13b})$$

$$\mathcal{L}_0 d_2 + \mathcal{L}_1 (\delta_1 \phi_r) = 0 \quad (\text{A13c})$$

$$\mathcal{L}_0 b_2 + \mathcal{L}_1 b_1 + \beta_0 \mathcal{L}_2 \phi_r = 0. \quad (\text{A13d})$$

The SC and (A13c) imply that $\delta_1 = 0$; so $d_1 = 0$. Thus (A13b) and (A13c) yield $c_2 = 0$ and $d_2 = \delta_2 \phi_r$. The SC and (A13d) give

$$\langle \phi_r \mathcal{L}_1 b_1 \rangle = -\beta_0 \langle \phi_r \mathcal{L}_2 \phi_r \rangle. \quad (\text{A14})$$

It is convenient to write b_1 in terms of two contributions, one of which is proportional to ϕ_r

$$b_1 = \beta_1 \phi_r + \Delta b_1 \quad (\text{A15})$$

where

$$\beta_1 = -\beta_0 \frac{\langle \phi_r \mathcal{L}_2 \phi_r \rangle}{\langle \phi_r \mathcal{L}_1 \phi_r \rangle}; \quad (\text{A16})$$

and Δb_1 is arbitrary except that it must satisfy $\langle \phi_r \mathcal{L}_1 \Delta b_1 \rangle = 0$. Note that this is precisely the condition that is satisfied by the otherwise arbitrary b_0 of the regular solution (see above), so choosing a different Δb_1 simply means adding a different amount of the regular solution onto the singular solution. Note also that a_0 and b_0 in the regular solution are independent, so that we may independently choose a_1 and Δb_1 in the singular solution to be whatever we like. We shall generally take Δb_1 to be zero unless otherwise indicated.

Finally at this order, (A13a) and (A13d) imply that

$$a_2 = \frac{-1}{i\lambda B^2} \{ (\mathcal{L}_0 - \lambda^2 B^2) b_1 + \beta_0 \mathcal{L}_1 \phi_r \} \quad (\text{A17a})$$

$$b_2 = -\mathcal{L}_0^{-1} (\mathcal{L}_1 b_1 + \beta_0 \mathcal{L}_2 \phi_r) + \beta_2 \phi_r. \quad (\text{A17b})$$

At the next order we find

$$-i\lambda B^2 (2a_3 + c_3) = (\mathcal{L}_0 - \lambda^2 B^2) b_2 + \mathcal{L}_1 b_1 + \beta_0 \mathcal{L}_2 \phi_r \quad (\text{A18a})$$

$$-2i\lambda B^2 c_3 = -\lambda^2 B^2 (\delta_2 \phi_r) \quad (\text{A18b})$$

$$2(\mathcal{L}_0 d_3 + \mathcal{L}_1 (\delta_2 \phi_r)) = \lambda^2 \beta_0 \mathcal{L}_1 \phi_r \quad (\text{A18c})$$

$$\begin{aligned} 2(\mathcal{L}_0 b_3 + \mathcal{L}_1 b_2 + \mathcal{L}_2 b_1 \\ + \beta_0 \mathcal{L}_3 \phi_r) = -\frac{1}{2} \lambda^2 \beta_0 \mathcal{L}_1 \phi_r \end{aligned} \quad (\text{A18d})$$

— we have used the fact that $c_1 = -i\lambda \beta_0 \phi_r$. From the SC and (A18c) one finds that $\delta_2 = \frac{1}{2} \lambda^2 \beta_0$; thus d_2 and c_3 are determined. Moreover, the SC and (A18d) determines β_2 and hence b_2 :

$$\begin{aligned} b_2 = -\frac{1}{4} \lambda^2 \beta_0 + \left\{ \langle \phi_r \mathcal{L}_1 \mathcal{L}_0^{-1} [(\beta_1 \mathcal{L}_1 + \beta_0 \mathcal{L}_2) \phi_r] \rangle \right. \\ \left. - \beta_1 \langle \phi_r \mathcal{L}_2 \phi_r \rangle - \beta_0 \langle \phi_r \mathcal{L}_3 \phi_r \rangle \right\} / \langle \phi_r \mathcal{L}_1 \phi_r \rangle. \end{aligned} \quad (\text{A19})$$

APPENDIX B: RECURSION FORMULAE FOR THE COEFFICIENTS OF THE GENERALIZED FROBENIUS SOLUTIONS

In this appendix it is demonstrated explicitly that the coefficients in each generalized Frobenius series (i.e., for the nonresonant case and for both regular and singular solutions in the resonant case) are expressible in terms of lower-order coefficients in the series, so that finding the series is a well-defined procedure once the first few terms are known.

Nonresonant Case

In this case $\sigma = 0$ and the c_n and d_n are all identically zero (see Appendix A). It then follows from equations (A3) that for $n \geq 0$

$$-i\lambda B^2(n+1)a_{n+1} = \sum_{m=0}^n \mathcal{L}_{n-m}b_m - \lambda^2 B^2b_n \quad (\text{B1})$$

and

$$(n+1) \sum_{m=0}^{n+1} \mathcal{L}_{n-m+1}b_m = i\lambda \sum_{m=0}^n \mathcal{L}_{n-m}a_m. \quad (\text{B2})$$

Since $\mathcal{L}_0$ is invertible in the nonresonant case, equation (B2) can be rewritten

$$b_{n+1} = -\mathcal{L}_0^{-1} \sum_{m=0}^n \mathcal{L}_{n-m+1}b_m + \frac{i\lambda}{n+1} \mathcal{L}_0^{-1} \sum_{m=0}^n \mathcal{L}_{n-m}a_m. \quad (\text{B3})$$

It is readily apparent from equations (B1) and (B3) that a_{n+1} and b_{n+1} are determined by the previous a_m 's and b_m 's: thus all the coefficients are determined once a_0 and b_0 (which are arbitrary) have been chosen.

Resonant Case: Regular Solution

As in the nonresonant case, $\sigma = 0$, the c_n and d_n are all identically zero and the coefficients satisfy equations (B1) and (B2). The difference now is that $\mathcal{L}_0^{-1}g$ is only defined if $\langle \phi_r, g \rangle = 0$ (the solvability condition SC) and even then is only defined up to an arbitrary additive multiple of ϕ_r . Clearly equation (B1) determines a_{n+1} in terms of $b_0, \dots, b_n$. It remains to be shown that b_{n+1} is determined by $a_0, \dots, a_n$ and $b_0, \dots, b_n$. Consider then equation (B2) for $n = 0$:

$$\mathcal{L}_0b_1 + \mathcal{L}_1b_0 = i\lambda \mathcal{L}_0a_0. \quad (\text{B4})$$

Using the solvability condition this implies that b_0 must satisfy $\langle \phi_r, \mathcal{L}_1b_0 \rangle = 0$ (but in fact is otherwise arbitrary) and that $b_1 = i\lambda a_0 - \mathcal{L}_0^{-1}\mathcal{L}_1b_0 + \beta_1\phi_r$, where β_1 has yet to be determined. We now proceed to prove that b_{n+1} is determined by the previous a 's and b 's using mathematical induction.

Proof. Suppose that $a_0, \dots, a_{N-1}$ and $b_0, \dots, b_{N-1}$ are already determined and that b_N is also determined except for the constant β_N :

$$b_N = \bar{b}_N(a_0, \dots, a_{N-1}, b_0, \dots, b_{N-1}) + \beta_N\phi_r. \quad (\text{B5})$$

Note that we know this to be true for $N = 1$, once a_0 and b_0 have been chosen. We shall demonstrate that β_N can be determined in terms of the a_m 's and b_m 's for $m < N$ and that equation (B5) with N replaced by $N + 1$ then

follows. The constant β_N is obtained from equation (B2) with $n = N$: it is useful first to rewrite this as

$$\begin{aligned} \mathcal{L}_0(b_{N+1} - \frac{i\lambda}{N+1}a_N) + \beta_N\mathcal{L}_1\phi_r = \\ -\mathcal{L}_1\bar{b}_N - \sum_{m=0}^{N-1} \mathcal{L}_{N-m+1}b_m \\ + \frac{i\lambda}{N+1} \sum_{m=0}^{N-1} \mathcal{L}_{N-m}a_m. \end{aligned} \quad (\text{B6})$$

Applying the SC to this determines β_N , in terms of the a_m and b_m for $m < N$ only. Note that β_N doesn't depend on a_N . Thus b_N is completely determined; and a_N can also be found from the previous b 's. Finally, having determined β_N to satisfy the SC, equation (B6) can be solved to find b_{N+1} in terms of a_m, b_m ($m = 0, \dots, N$) and an as yet undetermined constant β_{N+1} , in the form of equation (B5). This completes the proof by induction. QED

Resonant Case: Singular Solution

In this case $\sigma = -1$, and it follows from equations (A3), balancing both $x^{\sigma+n}$ terms and $x^{\sigma+n} \ln x$ terms that for $n \geq 1$

$$-i\lambda B^2(na_{n+1} + c_{n+1}) = -\lambda^2 B^2b_n + \sum_{m=0}^n \mathcal{L}_{n-m}b_m \quad (\text{B7a})$$

$$-i\lambda B^2nc_{n+1} = -\lambda^2 B^2d_n + \sum_{m=0}^n \mathcal{L}_{n-m}d_m \quad (\text{B7b})$$

$$\sum_{m=0}^{n+1} \mathcal{L}_{n-m+1}(nb_m + d_m) = i\lambda \sum_{m=0}^n \mathcal{L}_{n-m}a_m \quad (\text{B7c})$$

$$n \sum_{m=0}^{n+1} \mathcal{L}_{n-m+1}d_m = i\lambda \sum_{m=0}^n \mathcal{L}_{n-m}c_m. \quad (\text{B7d})$$

These may be re-arranged in the more convenient form

$$\begin{aligned} -i\lambda B^2n^2a_{n+1} &= -\lambda^2 B^2(nb_n - d_n) \\ &+ \sum_{m=0}^n \mathcal{L}_{n-m}(nb_m - d_m) \end{aligned} \quad (\text{B8a})$$

$$-i\lambda B^2nc_{n+1} = -\lambda^2 B^2d_n + \sum_{m=0}^n \mathcal{L}_{n-m}d_m \quad (\text{B8b})$$

$$n^2 \sum_{m=0}^{n+1} \mathcal{L}_{n-m+1}b_m = i\lambda \sum_{m=0}^n \mathcal{L}_{n-m}(na_m - c_m) \quad (\text{B8c})$$

$$n \sum_{m=0}^{n+1} \mathcal{L}_{n-m+1}d_m = i\lambda \sum_{m=0}^n \mathcal{L}_{n-m}c_m. \quad (\text{B8d})$$

In addition, it follows (see Appendix A) from balancing the lower-order terms in equations (A3) that

$$a_0 = c_0 = d_0 = 0, b_0 = \beta_0\phi_r, c_1 = -i\lambda b_0, d_1 = \delta_1\phi_r; \quad (\text{B9})$$

with β_0 and a_1 arbitrary. Also (which comes from applying the SC to (B8c))

$$b_1 = -\frac{\langle \phi_r, \mathcal{L}_2b_0 \rangle}{\langle \phi_r, \mathcal{L}_1\phi_r \rangle} \phi_r + \Delta b_1 \quad (\text{B10})$$

(Appendix A), where Δb_1 satisfies $\langle \phi_r \mathcal{L}_1 \Delta b_1 \rangle$ but is otherwise arbitrary. Equation (B8c) can now be solved for b_2 , except that b_2 contains an (as yet) undetermined additive multiple of the resonant eigenfunction. It is clear from (B8a) and (B8b) that a_{n+1} , c_{n+1} ($n \geq 1$) are determined once b_m and d_m ($m \leq n$) are known. Now equation (B8d) is essentially the same as equation (B2), with a_m and b_m replaced by c_m and d_m ; furthermore, we know a_0 , b_0 and c_0 and d_0 and d_1 is known to within an additive multiple of ϕ_r . Hence the induction proof of the previous section shows that d_{n+1} ($n \geq 0$) is determined once the c_m ($m \leq n$) are known. The same induction argument can be applied to equation (B8c) for the case of b_{n+1} , as $a_0, \dots, d_0, a_1, \dots, d_1$ can now be taken as known and b_2 is of the required form of being known except for an additive multiple of ϕ_r .

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