# Coupling of Fast and Alfvén Modes in Realistic Magnetospheric Geometries

## ANDREW N. WRIGHT<sup>1</sup>

Astronomy Unit, School of Mathematical Sciences, Queen Mary and Westfield College, London

The resonant coupling of linear fast and Alfvén modes is considered in a cold plasma permeated by a curl-free background magnetic field. The medium is assumed to possess an invariant coordinate (e.g., slab or axisymmetric geometry). We present the problem in terms of a set of orthogonal wave functions which describe the wave fields. Perturbations are Fourier analyzed along the invariant direction with wave number  $k_{\beta}$ , which is subsequently employed as a second expansion parameter with which to expand the governing equations. These equations are then solved using time-dependent perturbation theory, familiar in quantum mechanics. Our calculations provide generalisations to the results of previous authors, such as the excitation of resonant Alfvén waves and the damping of the fast mode. To test our novel formulation we compare our results with numerical solutions of the 'box' magnetosphere. For small azimuthal wave numbers (< 3 - 4) our lowest-order estimates of the cavity mode damping rates are in excellent agreement with previous calculations.

### 1. INTRODUCTION

In an inhomogeneous medium the coupling of one wave mode to another is often inevitable [Wright, 1990; Wright and Evans, 1991]. One of the best observed natural wave coupling phenomena occurs in the Earth's magnetosphere and is thought to be responsible for magnetic pulsations. Magnetic pulsations are standing Alfvén waves that have been resonantly excited on closed field lines deep within the magnetosphere. The force which drives the resonance is the magnetic pressure gradient of a fast mode, sometimes called the 'cavity' mode.

It is evident from the coherent and spatially localized nature of magnetic pulsations that the driving fast mode must have a regular oscillatory time dependence. Assuming a time dependence of  $exp[i\omega t]$ , resonant Alfvén wave excitation has been demonstrated in simplified 'box' systems incorporating a uniform magnetic field [Southwood, 1974; Chen and Hasegawa, 1974]. At first it was thought that the regular oscillatory behavior of the fast mode was due to convecting Kelvin-Helmholtz vortices at the magnetopause [Dungey, 1967]. More recently it has been suggested that the buffeting of the magnetosphere by the solar wind will excite fast cavity modes within the magnetosphere. Slight asymmetries in azimuth may lead to the resonant excitation of Alfvén waves on field lines where one of the natural Alfvén frequencies matches a cavity mode frequency [Kivelson and Southwood, 1985; Wright, 1992].

Over recent years, considerable effort has been made to improve modeling of magnetic pulsations. Some workers have studied resonant coupling in more realistic geometries than the box model but retained a time dependence  $e^{i\omega t}$ [Inhester, 1986; Allan et al., 1987; Cross, 1988; Mond et al., 1990]. Other investigations have relaxed this time dependence and consider both 'box' magnetospheres [Inhester, 1987; Zhu and Kivelson, 1988; Southwood and Kivelson,

Paper number 91JA02655. 0148-0227/92/91JA-02655\$05.00 1990] and alternative geometries [Allan et al., 1986a; Allan et al., 1986b; Lee and Lysak, 1989; Lee and Lysak, 1990; Wright, 1992]. With the exception of Southwood and Kivelson [1990] and Wright [1992] the latter work is largely numerical.

In this paper we extend the analysis presented by Wright [1992] to describe not only the growth of resonant Alfvén waves but also the damping of the fast cavity mode which loses energy to the Alfvén resonance [Zhu and Kivelson, 1988]. The paper is structured as follows: §2 introduces the magnetic coordinate system, presents the wave equations for the fast and Alfvén modes, and discusses the decoupled eigenmodes. The time evolution of the coefficients in a sum over eigenmodes is addressed in §3, where it is also shown that considerable simplification can be introduced by expanding these coefficients as a Taylor series about  $k_{\beta} = 0$ . In §4 our formalism is applied to modeling magnetic pulsations. The first-order Alfvén solution represents the resonant excitation of standing Alfvén waves, while the second-order fast mode solution represents damping of the original fast cavity mode. Our results are compared with exact solutions (for simple geometry) in §5, and we discuss the convergence of our series solution. §6 summarizes and concludes the paper.

## 2. GOVERNING EQUATIONS

The coordinate system used throughout this paper is an orthogonal curvilinear one based upon the magnetic geometry. We define three spatial coordinates  $(\alpha, \beta, \gamma)$  and let  $\hat{\gamma}$  be parallel to the local background magnetic field direction everywhere. The transverse coordinates  $(\alpha, \beta)$  are constant on any background line of force and are similar to Euler potentials or Clebsch variables. The background magnetic field is assumed to be solenoidal and irrotational, requiring

$$Bh_{\alpha}h_{\beta} = f(\alpha,\beta) \tag{1}$$

$$Bh_{\gamma} = g(\gamma) \tag{2}$$

where f and g are arbitrary functions of their arguments and the scale factors  $h_i$  are equal to  $\hat{1}/\nabla i$ , where  $i = \alpha, \beta, \gamma$ . A physical interpretation of the scale factors may be realized by noting that a real space element  $d\mathbf{r}$  is equal to  $\hat{\alpha}h_{\alpha}d\alpha + \hat{\beta}h_{\beta}d\beta + \hat{\gamma}h_{\gamma}d\gamma$ . These results are standard

<sup>&</sup>lt;sup>1</sup>Now at Department of Mathematical and Computational Sciences, University of St Andrews, St Andrews, Scotland

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properties of such a coordinate system [Davis and Snider, 1979]. Similar coordinate systems have facilitated earlier investigations of related problems [Singer et al., 1981; Southwood and Hughes, 1983; Walker, 1987; Wright, 1990; Wright and Smith, 1990]. We shall assume that the magnetospheric cavity is invariant in the  $\beta$  direction and has an arbitrary cross section in the  $(\alpha, \gamma)$  plane; see Figure 1. The boundary surface of the magnetospheric cavity may be defined in terms of the pair of functions  $\alpha_1(\gamma)$  and  $\alpha_2(\gamma)$ , where  $\alpha_1$ and  $\alpha_2$  are the lower and upper values of  $\alpha$  at which the line  $\gamma = \text{const}$  crosses the boundary. (Of course, these functions may be inverted to give  $\gamma_1(\alpha)$  and  $\gamma_2(\alpha)$ , which are the lower and upper values of  $\gamma$  at which the line  $\alpha = \text{const}$  crosses the boundary.)

In the cold plasma limit the entire wave field can be described in terms of the transverse plasma displacements  $\xi_{\alpha}$  and  $\xi_{\beta}$ . The linearized momentum and integrated induction equations may be combined to give the following inhomogeneous wave equations:

$$\frac{\partial}{\partial \gamma} \left( \frac{h_{\alpha}}{h_{\beta}h_{\gamma}} \cdot \frac{\partial}{\partial \gamma} (\xi_{\alpha}h_{\beta}B) \right) + \frac{\partial}{\partial \alpha} \left( \frac{h_{\gamma}}{h_{\alpha}h_{\beta}} \cdot \frac{\partial}{\partial \alpha} (\xi_{\alpha}h_{\beta}B) \right)$$
$$-h_{\alpha}h_{\gamma}\frac{B}{V^{2}} \cdot \frac{\partial^{2}\xi_{\alpha}}{\partial t^{2}} = -\frac{\partial}{\partial \alpha} \left( \frac{h_{\gamma}}{h_{\alpha}h_{\beta}} \cdot \frac{\partial}{\partial \beta} (\xi_{\beta}h_{\alpha}B) \right) \quad (3a)$$

$$\frac{\partial}{\partial \gamma} \left( \frac{h_{\beta}}{h_{\alpha}h_{\gamma}} \cdot \frac{\partial}{\partial \gamma} (\xi_{\beta}h_{\alpha}B) \right) - h_{\beta}h_{\gamma} \frac{B}{V^{2}} \cdot \frac{\partial^{2}\xi_{\beta}}{\partial t^{2}}$$
$$= -\frac{\partial}{\partial \beta} \left( \frac{h_{\gamma}}{h_{\alpha}h_{\beta}} \cdot \frac{\partial}{\partial \alpha} (\xi_{\alpha}h_{\beta}B) \right) - \frac{\partial}{\partial \beta} \left( \frac{h_{\gamma}}{h_{\alpha}h_{\beta}} \cdot \frac{\partial}{\partial \beta} (\xi_{\beta}h_{\alpha}B) \right)$$
(3b)

where V is the Alfvén speed. Evidently if  $\partial/\partial\beta = 0$ , the fast and Alfvén modes decouple, the fast mode being described by the plasma motion  $\xi_{\alpha}$  confined to a meridian plane while  $\xi_{\beta}$  represents axisymmetric toroidal Alfvén waves.

Some of the analysis techniques employed throughout this paper are simplest when applied to nondimensional quantities. For this reason we shall assume all quantities are nor-



Fig. 1. The cross section of the magnetospheric cavity in the  $(\alpha, \gamma)$  plane. The bold line represents the boundary of the magnetospheric cavity (the shaded region). Magnetic field lines are depicted with arrows to denote their direction, and are labeled by lines of constant  $\alpha$ . The remaining lines are contours of constant  $\gamma$ . The boundary of the cavity is described by the pair of functions  $\alpha_1(\gamma)$  and  $\alpha_2(\gamma)$ , which give the two values of  $\alpha$  for which the line  $\gamma$  =const crosses the boundary. These may be inverted to give the complementary functions  $\gamma_1(\alpha)$  and  $\gamma_2(\alpha)$ .

malized by representative values of the background medium. (For example, lengths can be divided by the equatorial standoff distance of the magnetopause. Magnetic fields, velocity fields, and plasma density can be measured relative to the field strength, Alfvén speed, and density at the nose of the magnetopause.)

We shall now show that when  $\partial/\partial\beta = 0$  there exist a complete orthogonal set of eigenfunctions for both the Alfvén and fast modes. These special functions provide a suitable basis with which the wave fields can be described during time-dependent coupling.

## Fast Cavity Eigenmodes

In order to calculate the fast, or cavity, eigenmodes we set  $\partial/\partial\beta = 0$  in (3a) and assume the mode  $\xi_{\alpha ij}(\alpha, \gamma)$  oscillates with an eigenfrequency  $\omega_{\alpha ij}$ . The functions  $\xi_{\alpha ij}$  are twodimensional eigenfunctions analogous to the displacement of a nonuniform drum skin. (Note that the plasma displacement in the fast eigenmodes is confined to planes  $\beta = \text{const}$  and is oriented perpendicular to B.) The two indexes ij reflect this two dimensionality and in a uniform medium would correspond to the number of nodes in  $\alpha$  and  $\gamma$  directions. In a nonuniform system such a simple nodal description is not always appropriate [Lee and Lysak, 1990]; nevertheless, we retain two indexes to describe the two degrees of freedom of the fast cavity eigenmodes. The eigenmode equation is

$$\frac{\partial}{\partial\gamma} \left( \frac{h_{\alpha}}{h_{\beta}h_{\gamma}} \cdot \frac{\partial}{\partial\gamma} \left( \xi_{\alpha i j} h_{\beta} B \right) \right) + \frac{\partial}{\partial\alpha} \left( \frac{h_{\gamma}}{h_{\alpha}h_{\beta}} \cdot \frac{\partial}{\partial\alpha} \left( \xi_{\alpha i j} h_{\beta} B \right) \right) \\ + \omega_{\alpha i j}^{2} h_{\alpha} h_{\gamma} \frac{B}{V^{2}} \cdot \xi_{\alpha i j} = 0$$
(4)

In Appendix A we show that under suitable boundary conditions (e.g.,  $\xi_{\alpha} = 0$  on the cavity boundary) any two different fast eigenmodes, say  $\xi_{\alpha ij}$  and  $\xi_{\alpha i'j'}$ , are orthogonal and may be normalized according to

$$\int_{\gamma_{min}}^{\gamma_{max}} \int_{\alpha_1(\gamma)}^{\alpha_2(\gamma)} \xi_{\alpha i j} \xi_{\alpha i' j'} h_{\alpha} h_{\beta} h_{\gamma} \frac{B^2}{V^2} d\alpha d\gamma = \delta_{i i'} \delta_{j j'} \quad (5)$$

The limits  $\gamma_{min}$  and  $\gamma_{max}$  represent the extreme  $\gamma$  values of the magnetospheric cavity.

Following Lee and Lysak [1990] we may write the  $\xi_{\alpha}$  component of any disturbance as a sum over the fast eigenmodes,

$$\xi_{\alpha}(\alpha,\beta,\gamma,t) = \sum_{ij} a_{\alpha ij}(\beta,t) \xi_{\alpha ij}(\alpha,\gamma)$$
(6)

When we write  $\xi_{\alpha}$  as a sum over the cavity eigenmodes the problem of describing the evolution of a fast mode disturbance becomes the problem of determining the time dependence of the set of fast mode coefficients  $\{a_{\alpha_{12}}\}$ .

## Alfvén Wave Eigenmodes

In contrast to the fast, or cavity, eigenmodes, which are analagous to the normal modes of a nonuniform drum skin, the Alfvén eigenmodes are similar to waves on a nonuniform string. Alfvén eigenmodes have received considerable attention in previous investigations of terrestrial magnetic pulsations [Dungey, 1954; Dungey, 1967; Radoski, 1967; Cummings et al., 1969; Warner and Orr, 1979; Singer et al., 1981; Southwood and Hughes, 1983] and waves in Jupiter's magnetosphere [Glassmeier et al., 1989; Smith and Wright, 1989; Wright and Smith, 1990]. The latter study notes that the eigenmode equation is of the Sturm-Liouville form when suitable boundary conditions are applied (e.g.,  $\xi_{\beta} = 0$  on the boundary of the magnetospheric cavity). The eigenmode equation for the *n*th eigenmode  $\xi_{\beta n}$ , which has a real eigenfrequency  $\omega_{\beta n}$ , is

$$\frac{\partial}{\partial\gamma}\left(\frac{h_{\beta}}{h_{\alpha}h_{\gamma}}\cdot\frac{\partial}{\partial\gamma}\left(\xi_{\beta n}h_{\alpha}B\right)\right)+\omega_{\beta n}^{2}h_{\beta}h_{\gamma}\frac{B}{V^{2}}\cdot\xi_{\beta n}=0\quad(7)$$

For each value of  $\alpha$  (i.e., each magnetic field line) there is a corresponding eigenmode equation like (7). Thus each magnetic field line has its own set of eigenmodes  $\{\xi_{\beta n}(\alpha, \gamma)\}$  and eigenfrequencies  $\{\omega_{\beta n}(\alpha)\}$ . The Alfvén eigenmodes can be thought of as a set of one-dimensional normal modes which vary with  $\gamma$ , the field aligned coordinate. (The coordinate  $\alpha$  enters as a parameter to define the field-line of interest.)

It is a well-known property of Sturm-Liouville systems that the eigenmodes form a complete orthogonal set [Morse and Feshbach, 1953]. On any given field line, two (normalised) modes will satisfy

$$\int_{\gamma_1(\alpha)}^{\gamma_2(\alpha)} \xi_{\beta n} \xi_{\beta n'} h_\alpha h_\beta h_\gamma \frac{B^2}{V^2} d\gamma = \delta_{nn'} \tag{8}$$

We are also able to synthesise any  $\xi_{\beta}$  disturbance from a suitable sum over the eigenmodes

$$\xi_{\beta}(\alpha,\beta,\gamma,t) = \sum_{n} a_{\beta n}(\alpha,\beta,t)\xi_{\beta n}(\alpha,\gamma)$$
(9)

Once again, our problem is how to determine the set of coefficients  $\{a_{\beta n}\}$  and is addressed in the next section.

#### 3. COUPLED COEFFICIENT EQUATIONS

To include coupling between the fast and Alfvén modes we need to introduce some dependence on the  $\beta$  coordinate in the wave fields. For the remainder of this paper we shall assume that all waves vary according to  $\exp[ik_{\beta}\beta]$ . Thus the  $\beta$  dependence of all the coefficients  $\{a_{\alpha i_j}\}$  and  $\{a_{\beta n}\}$ defined in (6) and (9) is also  $\exp[ik_{\beta}\beta]$ , although from here on we shall not write this dependence explicitly.

The equation governing the evolution of any fast mode coefficient can be determined by the following manipulation: Substitute the eigenfunction expansions for  $\xi_{\alpha}$  and  $\xi_{\beta}$  (equations (6) and (9)) into the inhomogeneous wave equation for  $\xi_{\alpha}$  (equation (3a)). The spatial derivatives on the left-hand side of (3a) may then be removed by recalling the definition of the eigenfunctions (equation (4)). Finally, the coefficient of any fast mode may be found by multiplying the resulting equation by the mode in question times  $h_{\beta}B$  and integrating over the  $(\alpha, \gamma)$  plane. Invoking the orthogonal property (5), we find that the coefficient  $a_{\alpha ij}$  is governed by

$$\frac{d^2 a_{\alpha i j}}{dt^2} + \omega_{\alpha i j}^2 a_{\alpha i j} = i k_\beta \int \int [\xi_{\alpha i j} h_\beta B$$
$$\times \frac{\partial}{\partial \alpha} \left( \frac{B h_\gamma}{h_\beta} \sum_n a_{\beta n} \xi_{\beta n} \right) d\alpha d\gamma] \equiv T_{ij}(t) \tag{10}$$

Evidently, when  $k_{\beta} = 0$ , the time dependence of any cavity mode coefficient,  $a_{\alpha ij}$ , is simply oscillatory with the natural frequency for that mode. When  $k_{\beta} \neq 0$ , any transverse Alfvén waves that are present will act as a driver for the fast mode coefficients, which is denoted by  $T_{ij}(t)$ . Of course, the fast mode (in addition to being driven by the Alfvén fields) will also drive the Alfvén wave equation (3b). In a similar fashion to the discussion above, we can also construct an equation governing the evolution of the Alfvén wave coefficient  $a_{\beta n}$  for any Alfvén wave eigenmode on any field line (i.e., any  $\alpha$ ): Substitute the series expansions for  $\xi_{\alpha}$ and  $\xi_{\beta}$  (equations (6) and (9)) into the inhomogeneous wave equation for  $\xi_{\beta}$  (equation (3b)). The field-aligned derivatives may be removed by recalling the definition of the eigenmodes (equation (7)). In order to select the coefficient of a particular mode, we multiply the resulting equation by the mode in question times  $h_{\alpha}B$  and integrate along the field line of interest. Finally, the orthonormality condition leads us to the following equation,

$$\frac{d^{2}a_{\beta n}}{dt^{2}} + \omega_{\beta n}^{2}a_{\beta n} = ik_{\beta}\int_{\gamma_{1}(\alpha)}^{\gamma_{2}(\alpha)}\frac{Bh_{\gamma}}{h_{\beta}}\xi_{\beta n}\frac{\partial}{\partial\alpha}\left(h_{\beta}B\sum_{ij}a_{\alpha ij}\xi_{\alpha ij}\right)d\gamma$$
$$-k_{\beta}^{2}\int_{\gamma_{1}(\alpha)}^{\gamma_{2}(\alpha)}\frac{B^{2}h_{\alpha}h_{\gamma}}{h_{\beta}}\xi_{\beta n}\left(\sum_{n}a_{\beta n}\xi_{\beta n}\right)d\gamma \equiv C_{n}(t) \quad (11)$$

The above equation shows how magnetic pressure gradients in the  $\hat{\beta}$  direction (due to plasma compression from both  $\xi_{\alpha}$ and  $\xi_{\beta}$  motions) will drive the transverse plasma displacements  $\xi_{\beta}$ .

The two sets of equations (10) and (11) represent a highly coupled system and are formidable to solve as they stand. A simplification which enables us to proceed further is to expand the sets of coefficients  $\{a_{\alpha ij}\}\$  and  $\{a_{\beta n}\}\$  as Taylor series about  $k_{\beta} = 0$ ,

$$a_{\alpha i j} = a_{\alpha i j}^{(0)} + k_{\beta} a_{\alpha i j}^{(1)} + k_{\beta}^2 a_{\alpha i j}^{(2)} + \dots \qquad (12a)$$

$$a_{\beta n} = a_{\beta n}^{(0)} + k_{\beta} a_{\beta n}^{(1)} + k_{\beta}^2 a_{\beta n}^{(2)} + \dots \qquad (12b)$$

We are employing  $k_\beta$  as a second expansion parameter, the first being the amplitude of the wave fields ( $\varepsilon$ ) which was used to linearise the governing equations (3a) and (3b) at the very beginning of the paper.

Some care must be taken when calculating the higherorder terms in the series (12). For example, we can only calculate the higher-order corrections (in  $k_{\beta}$ ) reliably up to an order  $\ln(\varepsilon)/\ln(k_{\beta})$ . Terms beyond this order are smaller than the second-order terms (in  $\varepsilon$ ) that were neglected in the initial linearisation of (3). For the remainder of this paper the 'order' of a term means the order in  $k_{\beta}$ , unless stated otherwise.

It is important to know how quickly the series in (12) converge and how many terms need to be calculated to determine the coefficients to a prescribed accuracy. We address these questions at the end of section 5.

Our analysis continues via the use of 'time-dependent perturbation theory' familiar in quantum mechanics [Schiff, 1968]: Substituting the expansion (12) into equation (10) for the coefficient  $a_{\alpha ij}$ , we find that the *m*th term in the expansion evolves according to

$$\frac{d^2 a_{\alpha i j}^{(m)}}{dt^2} + \omega_{\alpha i j}^2 a_{\alpha i j}^{(m)} = i \int \int [\xi_{\alpha i j} h_{\beta} B] \\ \times \frac{\partial}{\partial \alpha} \left( \frac{B h_{\gamma}}{h_{\beta}} \sum_{n} a_{\beta n}^{(m-1)} \xi_{\beta n} \right) d\alpha d\gamma = T_{i j}^{(m)}(t)$$
(13)

The above relation shows how  $a_{\alpha ij}^{(m)}(t)$  is completely determined by the set of lower-order Alfvén coefficients  $\{a_{\beta n}^{(m-1)}\}$ . Indeed, the lower-order (in  $k_{\beta}$ ) Alfvén modes can be thought of as drivers for the next order in the fast mode wave field.

In a similar fashion we can write down the mth order of equation (11),

$$\frac{d^{2}a_{\beta n}^{(m)}}{dt^{2}} + \omega_{\beta n}^{2}a_{\beta n}^{(m)} = i\int_{\gamma_{1}(\alpha)}^{\gamma_{2}(\alpha)}\frac{Bh_{\gamma}}{h_{\beta}}\xi_{\beta n}\frac{\partial}{\partial\alpha}\left(h_{\beta}B\sum_{ij}a_{\alpha ij}^{(m-1)}\xi_{\alpha ij}\right)d\gamma \\
- \int_{\gamma_{1}(\alpha)}^{\gamma_{2}(\alpha)}\frac{B^{2}h_{\alpha}h_{\gamma}}{h_{\beta}}\xi_{\beta n}\left(\sum_{n}a_{\beta n}^{(m-2)}\xi_{\beta n}\right)d\gamma \equiv C_{n}^{(m)}(t) \tag{14}$$

The above equation shows how a given term in the  $a_{\beta n}$  expansion is driven by both lower-order terms in that series and lower-order terms in the fast mode expansion series for  $a_{\alpha ij}$ .

The two equations (13) and (14) permit a much easier analysis than the coupled relations (10) and (11). We have manipulated the coupled equations into a hierarchy of decoupled equations in which each successive order is completely determined by lower orders.

#### 4. APPLICATION TO MAGNETIC PULSATIONS

In order to illustrate the behavior of the equations given in the previous section we shall apply our results to magnetic pulsations. Let us specify our initial conditions: For simplicity, we shall say that at t = 0 there is a single fast cavity mode (denoted by  $a_{\alpha c}$ ). All the other fast cavity mode coefficients  $\{a_{\alpha ij}; ij \neq c\}$  and transverse Alfvén coefficients  $\{a_{\beta n}\}$  are zero at t = 0, but may evolve at later times due to the influence of the cavity mode  $a_{\alpha c}$ .

## Zeroth-Order Cavity Mode Solution

Under the initial conditions stated above, the lowestorder (m = 0) cavity mode equation becomes a simple harmonic oscillator equation for  $a_{aij}^{(0)}$ . For modes  $ij \neq c$  the solution is simply  $a_{aij}^{(0)}(t) = 0$ , whilst the coefficient of our initial cavity mode (for t > 0) is governed by

$$a_{\alpha c}^{(0)} = a_{\alpha co} \sin(\omega_{\alpha c} t) \cdot e^{ik_{\beta}\beta}$$
(15)

where  $a_{aco}$  is the amplitude of the cavity mode. Thus to lowest order the cavity mode simply oscillates at its natural frequency. The solution (15) can be inserted into the m = 1equation (14) and will enable us to calculate the first-order Alfvén response of the field lines to the cavity mode  $a_{ac}^{(0)}$ .

#### First-Order Alfvén Mode Solution

The solution of this problem has been discussed by Wright [1992] for a variety of driving terms, including a steady harmonic driver like that in (15). (Readers should note that these calculations did not use the plasma displacement eigenfunctions, but the related magnetic field eigenmodes. Thus the  $a_{\beta n}$ 's used in this paper, although simply related to those of Wright [1992], are not identical.)

The first-order Alfvén wave coefficients evolve according to

$$\frac{d^2 a_{\beta n}^{(1)}}{dt^2} + \omega_{\beta n}^2 a_{\beta n}^{(1)} = i \int_{\gamma_1(\alpha)}^{\gamma_2(\alpha)} \frac{Bh_{\gamma}}{h_{\beta}} \xi_{\beta n} \frac{\partial}{\partial \alpha} \left( h_{\beta} B a_{\alpha c}^{(0)} \xi_{\alpha c} \right) d\gamma$$
$$\equiv C_n^{(1)}(t) = C_{no}^{(1)} \sin(\omega_{\alpha c} t) \qquad (16)$$

Since everything in the driving term is known, equation (16) completely determines the first-order Alfvén wave response. The parameter  $C_{no}^{(1)}$  represents how effectively the initial cavity mode can drive the first-order coefficient of the *n*th Alfvén wave eigenmode. Wright [1992] shows how the largest-amplitude Alfvén waves were excited on field lines where one of the natural Alfvén frequencies matched the cavity mode frequency. Let us assume that this resonant condition is satisfied on the field lines at  $\alpha = \alpha_r$  by the *r*th Alfvén mode (i.e.,  $\omega_{\beta r}(\alpha_r) = \pm \omega_{\alpha c}$ ). Since the eigenfrequencies are functions of  $\alpha$ , the resonant response will be localised in the  $\alpha$  direction.

We shall draw on the results of Wright [1992] which describe the growth of the Alfvén resonance. The amplitude of the resonant Alfvén wave eigenmode  $(a_{\beta\tau}^{(1)})$  grows linearly with time on the resonant field line  $(\alpha_{\tau})$ , and a peak begins to form. The width of the peak, expressed in terms of  $\alpha$ , can be related to the width in frequency via the function  $\omega_{\beta\tau}(\alpha)$ . Both of these widths were shown to be proportional to 1/t. Thus we are led to an approximate description of  $a_{\beta\tau}$ : Since the height of the peak of  $a_{\beta\tau}$  is proportional to t, and the width proportional to 1/t, we may approximate  $a_{\beta\tau}$  as a delta function after a few cycles. In Appendix B it is shown that the appropriate delta function is

$$a_{\beta r}^{(1)}(\alpha,t) = -\frac{C_{no}^{(1)}\pi}{2\omega_{\alpha c}} \cdot \cos(\omega_{\alpha c}t) \cdot \left|\frac{d\alpha}{d\omega_{\beta r}}\right| \cdot \delta(\alpha = \alpha_r) \quad (17)$$

Zhu and Kivelson [1988] show how this Alfvén resonance acts as a steady sink of cavity mode energy. We can see this feature in our model by considering the magnetic energy of the resonant field lines at  $\alpha = \alpha_r$ : In a crude fashion we could estimate the resonant magnetic energy by integrating  $b_{\beta}^2$  across the resonant peak. Since the height of the peak squared is proportional to  $t^2$  and the width is proportional to 1/t, we would expect the resonant magnetic field energy to be proportional to t, representing a steady drain on the cavity mode energy. In Appendix B we calculate the magnetic energy (per unit  $\beta$ ) quantitatively and find

$$W_{r}(t) \approx \frac{C_{n\sigma}^{(1)2}(\alpha_{r})tk_{\beta}^{2}}{4\mu_{0}\omega_{\alpha c}^{2}} \cdot \left|\frac{d\alpha}{d\omega_{\beta r}}\right|_{\alpha_{r}}$$
$$\times \int_{\gamma_{1}(\alpha_{r})}^{\gamma_{2}(\alpha_{r})} \frac{h_{\beta}}{h_{\alpha}h_{\gamma}} \cdot \left(\frac{\partial}{\partial\gamma}\left(\xi_{\beta r}h_{\alpha}B\right)\right)^{2}d\gamma \qquad (18)$$

The increasing energy of the Alfvén resonance damps the cavity mode [Zhu and Kivelson, 1988] and is described by a second-order correction to the cavity mode coefficients. (The first-order correction is identically zero if there is no seroth-order  $\xi_{\beta}$  field at t = 0.)

#### Second-Order Cavity Mode Solution

The second-order cavity mode coefficients  $a_{\alpha ij}^{(2)}$  are governed by equation (13) when m = 2,

$$\frac{d^2 a_{\alpha ij}^{(2)}}{dt^2} + \omega_{\alpha ij}^2 a_{\alpha ij}^{(2)} = i \int \int \xi_{\alpha ij} h_{\beta} B \cdot \frac{\partial}{\partial \alpha} \left( \frac{Bh_{\gamma}}{h_{\beta}} \sum_{n} a_{\beta n}^{(1)} \xi_{\beta n} \right) d\alpha d\gamma \equiv T_{ij}^{(2)}(t)$$
(19)

The coupling coefficient  $T_{ij}^{(2)}(t)$  may be integrated by parts to yield the alternative expression

$$T_{ij}^{(2)}(t) = i \int d\gamma \left\{ \left[ \xi_{\alpha ij} B^2 h_{\gamma} \sum_{n} a_{\beta n}^{(1)} \xi_{\beta n} \right]_{\alpha_1}^{\alpha_2} - \int \frac{\partial}{\partial \alpha} (\xi_{\alpha ij} h_{\beta} B) \cdot \frac{B h_{\gamma}}{h_{\beta}} \sum_{n} a_{\beta n}^{(1)} \xi_{\beta n} d\alpha \right\}$$
(20)

To simplify matters we can neglect all first-order Alfvén wave coefficients with the exception of the resonant coefficient  $a_{\beta r}^{(1)}$  defined in (17);  $\{a_{\beta n}^{(1)} = 0; n \neq r\}$ . Note that even though we need only consider a single nonsero Alfvén wave coefficient in (20), this coefficient will drive all of the cavity mode coefficients  $\{a_{\alpha ij}^{(2)}\}$  in principle. Indeed, a simple consideration of energy flux shows that this is inevitable: The second-order terms in the cavity mode expansion series must direct a net Poynting flux toward the resonant magnetic field lines where energy is being absorbed [Zhu and Kivelson, 1988]. It is not possible for a single cavity eigenmode to accomplish such energy transport, but a combination of different cavity eigenmodes with time-dependent coefficients can produce the required Poynting flux.

In this paper we shall only calculate a second-order correction to our initial cavity mode,  $a_{\alpha c}^{(2)}$ . The reason is that the driving function  $(T_c^{(2)})$  in (19) will oscillate with a frequency of  $\omega_{\alpha c}$  (see (17)), and so we would expect a largeamplitude resonant growth of the coefficient  $a_{\alpha c}^{(2)}$ . Since the growth of this coefficient will represent damping of our initial cavity mode, we will be able to estimate the decay rate (or damping time) of the cavity mode due to resonant Alfvén wave absorption. Calculation of these decay rates will allow us to compare our results with those of previous studies.

In order to calculate the  $a_{\alpha c}^{(2)}$  response we must first calculate the driving term  $T_c^{(2)}(t)$ . If we only consider the effect due to the resonant Alfvén coefficient  $a_{\beta r}$  (equation (17)), and assume that either  $\xi_{\alpha c}$  or  $\xi_{\beta r}$  is zero on the magnetospheric cavity boundary, then (20) simplifies to yield

$$T_{c}^{(2)}(t) \approx i \int_{\gamma_{1}(\alpha_{r})}^{\gamma_{2}(\alpha_{r})} \frac{\partial}{\partial \alpha} \left(\xi_{\alpha c} h_{\beta} B\right) \cdot \frac{Bh_{\gamma}}{h_{\beta}} \cdot \frac{C_{no}^{(1)} \pi}{2\omega_{\alpha c}} \cdot \left| \frac{d\alpha}{d\omega_{\beta r}} \right|$$

$$\times \xi_{\beta r} d\gamma \cos(\omega_{\alpha c} t) \equiv T_{co}^{(2)} \cos(\omega_{\alpha c} t)$$
(21)

After a few cycles of the above driver,  $a_{\alpha c}^{(2)}$  will be dominated by a secular term like

$$a_{\alpha c}^{(2)} \approx \frac{T_{co}^{(2)}t}{2\omega_{\alpha c}} \cdot \sin(\omega_{\alpha c}t)$$
 (22)

Substituting the solutions given in (15) and (22) for  $a_{\alpha c}^{(0)}$  and  $a_{\alpha c}^{(2)}$  into the series expansion (12) we have an oscillatory damped solution for the initial cavity mode. An estimate for the normalized damping rate  $\Gamma_c$  is found from

$$\Gamma_c \approx \frac{-k_{\beta}^2}{\omega_{\alpha c} |a_{\alpha c}^{(0)}|} \cdot \frac{d|a_{\alpha c}^{(2)}|}{dt}$$
(23)

Recalling the definitions of  $T_{co}^{(2)}$  (equation (21)),  $C_{no}^{(1)}$  (equation (16)) and  $a_{\alpha c}^{(0)}$  (equation (15)), the normalized damping rate can be calculated explicitly as

$$\Gamma_{c} \approx \frac{k_{\beta}^{2}\pi}{4\omega_{\alpha c}^{3}} \left| \frac{d\alpha}{d\omega_{\beta r}} \right|_{\alpha_{r}} \cdot \left[ \int_{\gamma_{1}(\alpha_{r})}^{\gamma_{2}(\alpha_{r})} \frac{\partial}{\partial \alpha} \left( \xi_{\alpha c} h_{\beta} B \right) \cdot \frac{Bh_{\gamma}}{h_{\beta}} \xi_{\beta r} d\gamma \right]^{2}$$

$$(24)$$

The above expression for the damping rate is determined by the geometry of the magnetic field and the decoupled eigenfunctions  $\xi_{\alpha c}$  and  $\xi_{\beta r}$ . These eigenfunctions are basic properties of the magnetospheric cavity and are easy to calculate numerically, even for complicated geometries. Of course, in simplified systems (like the box magnetosphere)  $\xi_{\beta r}$  is usually a trigonometric or hyperbolic function. In some models it is also possible to estimate the structure of the fast mode from WKB theory.

It is interesting to note that the secular nature of  $a_{\alpha c}^{(2)}$ means that for large t the first three terms in the series expansion (12) will not give an accurate approximation. This problem can be circumvented either by calculating more terms in the expansion series or by realising that we are at liberty to expand the fast mode oscillation frequency as a series in powers of  $k_{\beta}$  around the decoupled eigenfrequency  $\omega_{\alpha c}$ . Both of these methods are equivalent, the former representing a series expansion of the exponential function that the latter approach would introduce [Kevorkian and Cole, 1981].

## 5. DISCUSSION

In the previous section we showed how our novel formulation of wave coupling was able to reproduce qualitatively the behavior demonstrated in earlier studies (e.g., the growth of an Alfvén resonance and the damping of the cavity mode). In order to test our model quantitatively we shall calculate the normalized damping rate given in (24) for the box model magnetosphere and compare with the results of previous authors [*Zhu and Kivelson*, 1988].

Our results may be used to reproduce those of the box magnetosphere if we identify  $(\alpha, \beta, \gamma)$  with (x, y, z). (In this geometry,  $h_x = h_y = h_z = 1$ .) The magnetospheric cavity employed by Zhu and Kivelson has the plasmapause at x =0.1 and the magnetopause at x = 10.0. Moreover, they only consider modes which have a fundamental character along the field lines (in z), i.e., the Alfvén modes  $\xi_{y1}$  and the cavity modes  $\xi_{xi1}$ .

Zhu and Kivelson choose to normalize quantities relative to the background quantities at x = 1.0, except lengths which are multiplied by the wave number in the z direction. In addition, the magnetic field is uniform while the density varies as  $\rho = 1/x$ . Within this model the fundamental Alfvén frequency (normalized by the fundamental frequency at x = 1.0) is

$$\omega_{y1}(x) = x^{-1/2} \tag{25}$$

and consequently

$$\left|\frac{d\boldsymbol{x}}{d\boldsymbol{\omega}_{\mathbf{y}1}}\right| = 2\boldsymbol{x}^{3/2} \tag{26}$$

Finally, noting that z integration in (24) introduces a multiplicative factor of  $\pi/2$ , and that the normalisation condition (8) dictates  $\xi_{y1}(x) = \sqrt{2/\pi x}$ , we find the following expression for the damping rate,

TABLE 1. Parameters Characterizing the Resonant Damping of the Fast Mode in the Box Magnetosphere

Mode		$\omega_{xi1}(ZK)$	Ť <sub>T</sub>	$[d\xi_{xi1}/dx]_{x_{T}}$	F <sub>11</sub>	$F_{i1}(ZK)$
1	0.4228	0.415	5.594	0.0431	0.0444	0.0536
2	0.5402	0.521	3.427	0.0544	0.0377	0.0440
3	0.6612	0.633	2.287	0.0694	0.0358	0.0409

The table lists parameters governing the decay of fast cavity modes. All of the waves have a fundamental character along the background field lines. The first column specifies the variation of the fast cavity mode across the field lines (1, 2, 3); fundamental, second, or third harmonic). The second column gives the lowest-order estimate of the cavity mode frequency from our model, which is compared with the value found by Zhu and Kivelson [1988] (in the low  $\lambda$  or  $k_{\beta}$  limit) listed in the third column. The position of the Alfvén resonance is given in the fourth column, while the next column gives the lowest-order compression of the plasma at the resonant field line. The quantity  $F_{i1}$  is our estimate of the slope of the line in Figure 2 and may be compared with the same quantity deduced from Zhu and Kivelson's calculations (in the low  $\lambda$  or  $k_{\beta}$  limit) listed in the final column.

$$\Gamma_{i1} \approx \frac{k_{\beta}^2 \pi^2 \sqrt{x_r}}{4\omega_{xi1}^3} \cdot \left(\frac{d\xi_{xi1}}{dx}\right)_{x_r}^2 \equiv F_{i1} \cdot \frac{k_{\beta}^2}{\omega_{xi1}^{4/3}} \qquad (27)$$

Table 1 summarises our results following a simple numerical integration of the decoupled  $\xi_x$  eigenmode equation. (The shooting method was employed to solve for the first three eigenmodes and eigenfrequencies,  $\xi_{xi1}$  and  $\omega_{xi1}$ ; i = 1, 2, 3.) Once the cavity mode eigenfrequency has been determined, we can find the location of the resonant Alfvén waves  $x_r$  by requiring the fundamental Alfvén frequency (25) be equal to the cavity eigenfrequency. Once we know the position of the resonance, our numerical solution can be used to find  $d\xi_{xi1}/dx|_{x_r}$ . The values in Table 1 calculated by this method are accurate to at least three significant figures.

It is interesting to compare the oscillation frequencies for the cavity mode predicted by our analysis and that employed by *Zhu and Kivelson* [1988]. In the small  $k_{\beta}$  (or  $\lambda$ ) limit our 'decoupled' eigenfrequencies listed in Table agree with the real part of the 'damped' eigenfrequencies calculated by Zhu and Kivelson to better than 4%. The good agreement reinforces our expectation that to lowest order the fast mode oscillates as a decoupled eigenmode. Although Zhu and Kivelson do not tabulate any values for the location of the Alfvén resonance, we can estimate the position from their Figure 8. As far as it is possible to compare our predictions for the resonant field line location  $(x_r)$  with Zhu and Kivelson's, we find no apparent difference.

In order to compare our damping rates with those of Zhu and Kivelson we can equate our  $k_{\beta}$  with their  $\lambda$ , whilst our  $\Gamma$  is equivalent to  $\omega_i/\omega_r$  in their notation. Figure 2 is reproduced from Zhu and Kivelson [1988] and now includes three lines representing our estimates for the damping rates of the first three cavity modes. As one would expect, the agreement is very good when  $k_{\beta}$  (or  $\lambda$ ) is small. The slope of these lines is the parameter  $F_{i1}$  defined in equation (27) and listed in Table 1. The table also includes the Zhu and Kivelson estimate for  $F_{i1}$  (inferred from their Table 1), and these agree to 15% or better.

Our lowest-order estimates of the damping rates are probably slightly different to those of Zhu and Kivelson, in the small  $k_{\beta}$  or  $\lambda$  limit, because of our approximation of the resonant Alfvén wave coefficient as a delta function and the neglect of nonresonant Alfvén coefficients  $\{a_{\beta n}^{(1)} = 0; n \neq r\}$ . Inspection of Zhu and Kivelson's [1988] Figure 4b suggests



Fig. 2. The dependence of the cavity mode damping rate (for the fundamental (0), first (1), and second (2) harmonics) on the wave number  $k_{\beta}$  (or  $\lambda$  in Zhu and Kivelson's [1988] notation). The three curved lines represent the results of Zhu and Kivelson [1988], from which the original figure is taken. The three straight lines are the approximate lowest order estimates of the damping rates from our model, which is valid for small  $k_{\beta}$  (or  $\lambda$ ). This figure is adapted from Figure 6 of Zhu and Kivelson [1988].

that the delta function approximation is more suitable the higher the cavity mode harmonic (since the Alfvér fields occupy a thinner peak with little disturbance away from the resonance). Indeed, we find that the discrepancy between our damping rates and those of Zhu and Kivelson is less for the higher harmonic cavity modes.

The discrepancy between our damping estimates and those of *Zhu and Kivelson* beyond 0.25-0.35 on the horisontal axis of Figure 2 can be attributed to our neglect of higher-order terms in the expansions (12). Including higherorder corrections should extend the range of agreement between the two calculations.

The interval 0.25-0.35 on the horisontal axis of Figure 2 can be converted into a  $k_\beta$  interval of 0.33-0.38, which is equivalent to an asimuthal wave number in an axisymmetric geometry of 3-4. When the wave number becomes this large, our lowest-order solution begins to deviate from the exact solutions of *Zhu and Kivelson*. Hence our lowest-order solution (including the delta function approximation and the neglect of nonresonant Alfvén waves) is correct to 15% or better for asimuthal wave numbers less than 3-4.

Now consider the convergence of the series expansions for the fast and Alfvén coefficients in (12). These expansions are Taylor series and will always give a good approximation to the true coefficients if sufficiently many terms are calculated. In the present paper we have calculated the first three terms. To calculate higher-order terms is harder work, and if we wish to go beyond an order  $|\ln(\varepsilon)/\ln(k_{\beta})|$  nonlinear terms must be included in the governing equations. Evidently it is desirable to estimate how well a truncated series solution represents the true coefficient.

A series is convergent if the magnitude of each term in the series is less than that of the previous term. Since we have not derived general expressions for the coefficients here, we cannot make firm statements about the ultimate convergence of the series. However, we can compare the magnitudes of the first few terms and get a good idea of the accuracy of a truncated series solution.

The condition that  $k_{\beta}^2 a_{\alpha ij}^{(2)}/a_{\alpha ij}^{(0)} < 1$  can be reformulated using (15), (21), (22), and (24) to yield  $\omega_{\alpha c} t \Gamma_c < 1$ . The latter inequality reflects the fact that the exponential decay of the cavity mode amplitude  $(\exp[-\Gamma_c t])$  is well represented by the series solution  $([1 - \Gamma_c t])$  for small t only. (See the end of §4, also.) One might be inclined to think that a truncated  $a_{\alpha ij}$  expansion is always a good approximation regardless of the value of  $k_{\beta}$ . Clearly this is not the case according to Figure 2. The resolution of the problem lies in realising that although  $k_{\beta}^2 a_{\alpha ij}^{(2)} a_{\alpha ij}^{(2)} < 1$  when  $\omega_{\alpha c} t \Gamma_c < 1$ , it is not necessarily true that  $k_{\beta}^4 a_{\alpha ij}^{(4)} / k_{\beta}^2 a_{\alpha ij}^{(2)} < 1$ . The latter condition is sensitive to the value of  $k_{\beta}$ .

Earlier in this section it was shown that the truncated series solution (up to and including  $k_{\beta}^2 a_{\alpha ij}^{(2)}$ ) gives a good approximation for asimuthal wave numbers less than 4. For larger wave numbers we must include higher order terms to obtain a good approximation: The sets of coefficients  $\{a_{\beta n}^{(1)}\}$ and  $\{a_{\alpha ij}^{(2)}\}$  can be used to solve for  $\{a_{\beta n}^{(3)}\}$ , which in turn can be used to calculate  $\{a_{\alpha ij}^{(4)}\}$ . Including the fourth-order term would increase the range of  $k_{\beta}$  in Figure 2 over which our solution is a good representation of the exact solution. (We would model the beginning of the curve as a parabola rather than a straight line.)

## 6. CONCLUDING REMARKS

In this paper we have developed a novel model for describing the coupling of fast and Alfvén waves in a cold, inhomogeneous, current-free background medium. Given suitable boundary conditions it is possible to calculate a set of orthogonal eigenfunctions which may be used to represent any wave fields via a weighted sum. Employing techniques from quantum mechanics (time-dependent perturbation theory) we can deduce equations governing the coefficients in such a sum. These coefficients are highly coupled and difficult to solve for. If the medium has an invariant direction, we may Fourier analyse the wave fields in the invariant coordinate, e.g.,  $\exp[ik_{\beta}\beta]$ . If the coefficients are expanded as a Taylor series in  $k_{\beta}$ , the equations simplify to a decoupled hierarchy. In §4 we demonstrate how such a formulation can describe the excitation of an Alfvén resonance, like those in magnetic pulsations (see Wright [1992] also). Moreover we are able to examine the effect of the Alfvén resonance, and we find (as one would expect) that energy is removed from the fast mode. The qualitative features of our model are identical with the type of behavior described in previous investigations. (Note that we have not restricted ourselves to any particular magnetic field geometry. Thus our conclusions are good for any two-dimensional solenoidal background magnetic field.)

It should be borne in mind that the eigenmode description of wave coupling developed in this paper, and summarized in equations (12)-(14), gives a complete and accurate account of the wave fields for all times. The fact that our results are in good, but not exact, agreement with previous studies (in the limit  $k_{\beta} \rightarrow 0$ ) can be attributed to two simplifications introduced in §4. The first simplification was to approximate the Alfvén response of the medium to simply a resonant response, and neglect all other nonresonant coefficients,  $\{a_{\beta n}^{(1)} = 0; n \neq r\}$ . The second approximation was to treat the resonant Alfvén coefficient as a delta function (equation (17)). The fact that these simplifications introduce a discrepancy between our results and those of previous workers of 15% or less (for azimuthal wave numbers less than 3-4) suggests that we have employed reasonable approximations. If our equations (12)-(14) are solved without simplification, we would expect to reproduce previous results exactly.

#### APPENDIX A

In this appendix we shall investigate the orthogonality of the fast cavity eigenmodes. One eigenmode  $(\xi_{\alpha ij})$  and eigenfrequency  $(\omega_{\alpha ij})$  are defined by equation (4). Let us consider a second eigenfunction  $(\xi_{\alpha i'j'})$  which has a different eigenfrequency  $(\omega_{\alpha i'j'})$ , defined by

$$\frac{\partial}{\partial \gamma} \left( \frac{h_{\alpha}}{h_{\beta}h_{\gamma}} \cdot \frac{\partial}{\partial \gamma} \left( \xi_{\alpha i'j'} h_{\beta} B \right) \right) \\
+ \frac{\partial}{\partial \alpha} \left( \frac{h_{\gamma}}{h_{\alpha}h_{\beta}} \cdot \frac{\partial}{\partial \alpha} \left( \xi_{\alpha i'j'} h_{\beta} B \right) \right) + \omega_{\alpha i'j'}^2 h_{\alpha} h_{\gamma} \frac{B}{V^2} \cdot \xi_{\alpha i'j'} = 0 \tag{A1}$$

If we take equation (4) and multiply it by  $\xi_{\alpha i'j'}h_{\beta}B$ , then subtract from it equation (A1) times  $\xi_{\alpha ij}h_{\beta}B$ , the result may be written

$$\frac{\partial}{\partial \gamma} \left[ \xi_{\alpha i'j'} \cdot \frac{Bh_{\alpha}}{h_{\gamma}} \cdot \frac{\partial}{\partial \gamma} (\xi_{\alpha ij} h_{\beta} B) -\xi_{\alpha ij} \cdot \frac{Bh_{\alpha}}{h_{\gamma}} \cdot \frac{\partial}{\partial \gamma} (\xi_{\alpha i'j'} h_{\beta} B) \right] \\ + \frac{\partial}{\partial \alpha} \left[ \xi_{\alpha i'j'} \cdot \frac{Bh_{\gamma}}{h_{\alpha}} \cdot \frac{\partial}{\partial \alpha} (\xi_{\alpha ij} h_{\beta} B) -\xi_{\alpha ij} \cdot \frac{Bh_{\gamma}}{h_{\alpha}} \cdot \frac{\partial}{\partial \alpha} (\xi_{\alpha i'j'} h_{\beta} B) \right] \\ + \xi_{\alpha ij} \xi_{\alpha i'j'} h_{\alpha} h_{\beta} h_{\gamma} \cdot \frac{B^2}{V^2} \cdot (\omega_{\alpha ij}^2 - \omega_{\alpha i'j'}^2) = 0$$
(A2)

If we integrate the above equation over the cross section of the magnetospheric cavity (see Figure 1), we find

$$\begin{split} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \left[ \xi_{\alpha i'j'} \cdot \frac{Bh_{\alpha}}{\frac{1}{\gamma_{\gamma}}} \cdot \frac{\partial}{\partial \gamma} (\xi_{\alpha ij} h_{\beta} B) \right. \\ \left. -\xi_{\alpha ij} \cdot \frac{Bh_{\alpha}}{h_{\gamma}} \cdot \frac{\partial}{\partial \gamma} (\xi_{\alpha i'j'} h_{\beta} B) \right]_{\gamma_1(\alpha)}^{\gamma_2(\alpha)} \\ \left. + \int_{\gamma_{min}}^{\gamma_{max}} d\gamma \left[ \xi_{\alpha i'j'} \cdot \frac{Bh_{\gamma}}{h_{\alpha}} \cdot \frac{\partial}{\partial \alpha} (\xi_{\alpha ij} h_{\beta} B) \right. \\ \left. -\xi_{\alpha ij} \cdot \frac{Bh_{\gamma}}{h_{\alpha}} \cdot \frac{\partial}{\partial \alpha} (\xi_{\alpha i'j'} h_{\beta} B) \right]_{\alpha_1(\gamma)}^{\alpha_2(\gamma)} \end{split}$$

$$+(\omega_{\alpha i j}^{2}-\omega_{\alpha i' j'}^{2})\int_{\gamma_{min}}^{\gamma_{max}}\int_{\alpha_{1}(\gamma)}^{\alpha_{2}(\gamma)}\xi_{\alpha i j}\xi_{\alpha i' j'}$$
$$\times h_{\alpha}h_{\beta}h_{\gamma}\cdot\frac{B^{2}}{V^{2}}d\alpha d\gamma=0 \qquad (A3)$$

Recall that  $\alpha_1(\gamma)$  and  $\alpha_2(\gamma)$  are the upper and lower values of  $\alpha$  for which the line  $\gamma =$ const crosses the boundary of the magnetospheric cavity. (These may be inverted to give the complementary functions  $\gamma_1(\alpha)$  and  $\gamma_2(\alpha)$ .) The upper and lower bounds of the coordinates on the boundary are  $[\alpha_{min}, \alpha_{max}]$  and  $[\gamma_{min}, \gamma_{max}]$ . To arrive at the orthogonal relation quoted in equation (5), we need to consider the boundary conditions on the surface of the magnetospheric cavity. For example, if the decoupled eigenmode plasma displacement  $\xi_{\alpha ij}$  vanishes on the boundary, then the first two integrals in equation (A3) are zero, and so we have proved (5). Such a boundary condition would be appropriate if the plasma density in the boundary was much larger than the density throughout the cavity (e.g., an ionospheric boundary).

Alternatively, the first integral in (A3) will vanish if  $(\partial/\partial\gamma)(\xi_{\alpha ij}h_{\beta}B) = 0$  on the cavity boundary. This corresponds to the eigenmode having zero  $b_{\alpha}$  perturbation on the boundary. The second integral is also zero if  $(\partial/\partial\alpha)(\xi_{\alpha ij},h_{\beta}B) = 0$  on the boundary, corresponding to zero compressional field perturbations there. Of course, it is possible to have a suitable combination of different boundary and still arrive at the orthogonal property given in (5).

## APPENDIX B

In this appendix we consider the growth of the resonant Alfvén wave coefficient according to equation (16). This problem has already been discussed by *Wright* [1992], where it is shown that the time dependence of the resonant mode  $(a_{\beta r})$  for  $t \ge 0$  is

$$a_{\beta r}(\alpha, t) = \frac{C_{no}^{(1)}(\alpha)}{\omega_{\alpha c}^2 - \omega_{\beta r}^2} \cdot \left[-\sin(\omega_{\alpha c} t) + \frac{\omega_{\alpha c}}{\omega_{\beta r}} \cdot \sin(\omega_{\beta r} t)\right] \qquad \omega_{\beta r} \neq \omega_{\alpha c} \quad (B1a)$$

$$a_{\beta r}(\alpha, t) = C_{no}^{(1)}(\alpha_r) \left[ \frac{\sin(\omega_{ac} t)}{2\omega_{ac}^2} - \frac{t}{2\omega_{ac}} \cdot \cos(\omega_{ac} t) \right] \qquad \omega_{\beta r} = \omega_{ac} \quad (B1b)$$

Note that the coordinate  $\alpha$  enters the equations as a parameter which defines the field line of interest. In this sense,  $a_{\beta r}$  and  $\omega_{\beta r}$  are functions of  $\alpha$ . On nonresonant field lines  $(\alpha \neq \alpha_r : \omega_{\beta r} \neq \omega_{\alpha c})$ , (B1a) is appropriate. As we approach the resonant field line  $(\alpha \rightarrow \alpha_r : \omega_{\beta r} \rightarrow \omega_{\alpha c})$ , (B1a) may be shown to yield the second relation (B1b), which exhibits the secular growth familiar in resonant systems.

In the main text we discuss the behavior of  $a_{\beta r}$  qualitatively; the coefficient may be approximated by a delta function at the resonant field line. This appendix determines the appropriate magnitude of the delta function by integrating  $a_{\beta r}(\alpha)$  across the resonance located at  $\alpha = \alpha_r$ . Suppose we concentrate on the interval  $[\alpha_-, \alpha_+]$  which contains  $\alpha_r$ . We wish to evaluate

$$\int_{\alpha_{-}}^{\alpha_{+}} a_{\beta r} d\alpha = \int_{\omega_{\beta r-}}^{\omega_{\beta r+}} \left| \frac{d\alpha}{d\omega_{\beta r}} \right| \cdot a_{\beta r} d\omega_{\beta r} \qquad (B2)$$

In the second relation we have changed variables from  $\alpha$  to  $\omega_{\beta r}$ , since  $a_{\beta r}$  is foremost a function of  $\omega_{\beta r}$ . The integration interval  $[\omega_{\beta r-}, \omega_{\beta r+}]$  contains the resonant frequency  $\omega_{\alpha c}$  (which, of course, occurs at  $\alpha = \alpha_r$ ).

To perform a detailed evaluation of (B2), it is convenient to change variables once more to a frequency  $\Delta$  centered on the resonance;  $\omega_{\beta r} = \Delta + \omega_{\alpha c}$  with limits  $[\omega_{\beta r-}, \omega_{\beta r+}] \rightarrow$  $[-\Delta_o, +\Delta_o]$ . Employing this substitution and the definition of  $a_{\beta r}$  given in (B1) we find the integral in (B2) may be written

$$\int_{-\Delta_{\bullet}}^{+\Delta_{\bullet}} \frac{-C_{n\sigma}^{(1)}}{2\omega_{\alpha c}\Delta + \Delta^{2}} \cdot \left[-\sin(\omega_{\alpha c}t) + \frac{1}{\Delta/\omega_{\alpha c} + 1} \cdot \sin(\omega_{\alpha c}t + \Delta t)\right] \cdot \left|\frac{d\alpha}{d\omega_{\beta r}}\right| d\Delta \qquad (B3)$$

The peak in  $a_{\beta r}$  has a full width (at half maximum) in the parameter  $\Delta$  of about 8/t [Wright, 1992]. Thus, after a few cycles the  $\Delta$  interval we need to consider will become very small. By expanding the integrand in powers of  $\Delta$  and only retaining the lowest-order terms the integral may be written (following trigonometric manipulation)

$$\int_{-\Delta_{o}}^{+\Delta_{o}} \frac{-C_{no}^{(1)}}{2\omega_{ac}\Delta} \cdot \left[-\sin(\omega_{ac}t) + \sin(\omega_{ac}t)\cos(\Delta t) + \cos(\omega_{ac}t)\sin(\Delta t)\right] \cdot \left|\frac{d\alpha}{d\omega_{\beta r}}\right| d\Delta \qquad (B4)$$

Since the peak of  $a_{\beta r}$  is so highly localised, we may neglect the variation of  $C_{no}^{(1)}(\alpha)$  and  $d\alpha/d\omega_{\beta r}$  across the range of integration. The first two terms in the square brackets above will integrate to give sero. The third term gives a factor proportional to the sine integral [see Gradshteyn and Ryzhik, 1980, equation 3.7211]. The latter integration introduces a factor of  $\pi$ , so that the appropriate delta function to describe  $a_{\beta r}$  is

$$a_{\beta r}(\alpha,t) \approx -\left|\frac{d\alpha}{d\omega_{\beta r}}\right| \cdot \frac{C_{no}^{(1)}\pi}{2\omega_{\alpha c}} \cdot \cos(\omega_{\alpha c}t) \cdot \delta(\alpha = \alpha_r)$$
 (B5)

In §4 we also discussed the magnetic energy of the resonant field lines. The resonant Alfvén wave magnetic field perturbation is defined (via the integrated induction equation) in terms of the displacement eigenmode  $\xi_{\beta r}$ ,

$$b_{\beta r} = \frac{1}{h_{\alpha}h_{\gamma}} \cdot \frac{\partial}{\partial \gamma} (\xi_{\beta r} h_{\alpha} B)$$
(B6)

By integrating over the volume surrounding the resonance, the resonant magnetic energy  $(W_r)$  may be written

$$W_{r} = k_{\beta}^{2} \int \int \int_{\alpha_{-}}^{\alpha_{+}} \frac{a_{\beta r}^{2} b_{\beta r}^{2}}{2\mu_{0}} \cdot h_{\alpha} h_{\beta} h_{\gamma} d\alpha d\beta d\gamma \qquad (B7)$$

Given the dependence of  $a_{\beta\tau}$  on  $\beta$ , integration with respect to  $\beta$  is straightforward. In particular, if we are interested in the magnetic energy per unit  $\beta$ , this integration introduces a multiplicative factor of  $2/\pi$ . Once again, we can appeal to the fact that the resonance becomes highly localised in  $\alpha$ , which allows us to neglect the dependence of background quantities on  $\alpha$ . As a result we only need perform the  $\gamma$ integration along the resonant field line. Combining these simplifications we find that the magnetic energy (per unit  $\beta$ ) is given by

$$W_{\tau} = k_{\beta}^2 I(\xi_{\beta\tau}, \alpha_{\tau}) \times \int_{-\Delta_o}^{+\Delta_o} a_{\beta\tau}^2 \cdot \left| \frac{d\alpha}{d\omega_{\beta\tau}} \right| d\Delta \qquad (B8a)$$

where we have employed the same substitution as used in (B3). The integral  $I(\xi_{\beta\tau}, \alpha_{\tau})$  is defined as

$$I(\xi_{\beta r}, \alpha_r) = \frac{1}{\pi \mu_0} \cdot \int_{\gamma_1(\alpha_r)}^{\gamma_2(\alpha_r)} \frac{h_\beta}{h_\alpha h_\gamma} \cdot \left(\frac{\partial}{\partial \gamma} \left(\xi_{\beta r} h_\alpha B\right)\right)^2 d\gamma$$
(B8b)

For the moment we shall concentrate on the integral over  $\Delta$  in (B8a). From the definition of  $a_{\beta r}$  in (B1) we find that to lowest order in  $\Delta$ ,

$$a_{\beta r}^{2} = \frac{C_{no}^{(1)2}}{4\omega_{\alpha c}^{2}\Delta^{2}} \cdot \left[\sin^{2}(\omega_{\alpha c}t) + \sin^{2}(\omega_{\alpha c}t + \Delta t) -2\sin(\omega_{\alpha c}t)\sin(\omega_{\alpha c}t + \Delta t)\right]$$
(B9)

Expanding  $\sin(\omega_{\alpha c}t + \Delta t)$  with trigonometric identities we can multiply out the contents of the square brackets in (B9) to give

$$a_{\beta r}^{2} = \frac{C_{no}^{(1)2}}{4\omega_{\alpha c}^{2}\Delta^{2}} \cdot \left[\sin^{2}(\omega_{\alpha c}t)\left(1 + \cos^{2}(\Delta t) - 2\cos(\Delta t)\right) + \cos^{2}(\omega_{\alpha c}t)\sin^{2}(\Delta t)\right]$$
(B10)

where we have omitted odd functions of  $\Delta$  (since they will integrate to give zero). The terms in square brackets from (B10) may be manipulated to give

$$\begin{bmatrix} \sin^2(\omega_{\alpha c}t) \left( 2(1 - \cos(\Delta t)) - \sin^2(\Delta t) \right) \\ + \cos^2(\omega_{\alpha c}t) \sin^2(\Delta t) \end{bmatrix} \equiv \begin{bmatrix} 4\sin^2(\omega_{\alpha c}t) \sin^2(\Delta t/2) \\ + \sin^2(\Delta t)(\cos^2(\omega_{\alpha c}t) - \sin^2(\omega_{\alpha c}t)) \end{bmatrix}$$
(B11)

Some final trigonometric manipulations and the introduction of the variable  $\Delta' = \Delta t$  yields the integral over  $\Delta$  from (B8a) in the form

$$\frac{C_{no}^{(1)2}(\alpha_{r})}{4\omega_{\alpha c}^{2}} \cdot \int_{-\Delta_{o}^{\prime}t}^{+\Delta_{o}^{\prime}t} \left[ 2\left(1 - \cos(2\omega_{\alpha c}t)\right) \cdot \frac{\sin^{2}(\Delta^{\prime}/2)}{\Delta^{\prime 2}} \cdot t + \cos(2\omega_{\alpha c}t) \cdot \frac{\sin^{2}(\Delta^{\prime})}{\Delta^{\prime 2}} \cdot t \right] \cdot \left| \frac{d\alpha}{d\omega_{\beta r}} \right| d\Delta^{\prime}$$
(B12)

Employing standard results from *Gradshteyn and Ryzhik* [1980, equation 3.7413] we arrive at our final expression for the magnetic energy of the Alfvén resonance,

$$W_{r} \approx \frac{C_{no}^{(1)2}(\alpha_{r})\pi tk_{\beta}^{2}}{4\mu_{0}\omega_{\alpha c}^{2}} \cdot \left|\frac{d\alpha}{d\omega_{\beta r}}\right|_{\alpha_{r}}$$
$$\times \int_{\gamma_{min}(\alpha_{r})}^{\gamma_{max}(\alpha_{r})} \frac{h_{\beta}}{h_{\alpha}h_{\gamma}} \cdot \left(\frac{\partial}{\partial\gamma}(\xi_{\beta r}h_{\alpha}B)\right)^{2} d\gamma \quad (B13)$$

This relation demonstrates how the magnetic energy of the Alfvén fields at the resonance increases steadily with time and acts as a sink for the cavity mode energy.

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A. Wright, Department of Mathematical and Computational Sciences, University of St. Andrews, St. Andrews KY16 9SS, Scotland.

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