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# Asymptotic and Time-Dependent Solutions of Magnetic Pulsations in Realistic Magnetic Field Geometries

ANDREW N. WRIGHT[1]

*Astronomy Unit, School of Mathematical Sciences, Queen Mary and Westfield College, London*

The coupling of compressional and Alfvén modes is important in solar, magnetospheric, and laboratory plasmas. We investigate such coupling within a general framework, but concentrate upon applying our results to the excitation of Alfvén waves in the magnetosphere. The Alfvénic response of an arbitrary (curl-free) magnetic field and ideal cold plasma is considered, and the Alfvén wave equation is shown to be driven by perturbation magnetic pressure gradients aligned with the Alfvén fields. The efficiency with which any particular standing Alfvén wave is excited is governed by an overlap integral along the background field line of the Alfvén mode and the perturbation magnetic pressure gradient. Three types of compressional driving function are investigated which produce diverse behavior and may excite resonant and nonresonant Alfvén waves. These waves oscillate with either the natural Alfvén frequency or the frequency of the driver. Finally we turn our attention to the long-term, or asymptotic, state of the fields. A nonlinear solution is derived which is a generalization of earlier work. We find that the standing Alfvén waves will be confined to thin layers.

## 1. INTRODUCTION

Wave coupling is a natural feature of most inhomogeneous media and has been the subject of much research. Coupling is thought to be important for heating laboratory and solar plasmas, and also responsible for exciting magnetic pulsations in planetary magnetospheres. In this paper we shall focus our attention on the latter example, although our results are of general interest: Magnetic pulsations are thought to be standing Alfvén waves that have become established on closed field lines deep within the Earth's magnetosphere.

One mechanism by which pulsations are excited is through asymmetries in fast cavity modes, resulting from the buffeting of the magnetopause by the solar wind [*Kivelson and Southwood*, 1985]. Alternatively, the persistent convection of Kelvin-Helmholtz vortices along the magnetopause can excite an oscillatory (although spatially evanescent) fast mode which can drive field line resonances [*Southwood*, 1974; *Chen and Hasegawa*, 1974]. Recently it has been suggested that the motion of reconnected flux along the magnetopause or the collision of dense plasma clouds with the magnetopause can also excite standing Alfvén waves [*Southwood and Kivelson*, 1990; *Lühr et al.*, 1990]. Within the formalism we develop here, each of these mechanisms can be represented as a driving term in a quite general Alfvén wave equation. Our analysis has two advantages over previous models; first, we are not restricted to simple magnetic geometries; and second, we do not confine ourselves to solutions with a harmonic time-dependence [*Inhester*, 1986; *Kivelson and Southwood*, 1986; *Mond et al.*, 1990].

Our calculations are tractable through the use of time-dependent perturbation theory, familiar in quantum mechanics [*Schiff*, 1968]. Such an approach is based upon the normal modes of the system. The results demonstrate the resonant excitation of Alfvén waves in a quite arbitrary magnetic field geometry. This calculation is instructive for calculating the growth rate of a magnetic pulsation. However, the asymptotic state of a magnetic pulsation may not be described very well by this model because it neglects ionospheric dissipation and also because the width of the resonant sheet of field lines becomes very small. Recently it has been suggested that Hall currents will be important in the final state [*Rajaram and Venkatesan*, 1990]. In this paper we give an alternative solution, which is a generalization of Dungey's highly asymmetric poloidal mode [*Dungey*, 1954, 1967].

The paper is structured as follows: Section 2 describes the coordinate system used throughout the paper along with the linearized cold plasma equations; section 3 analyses the equations presented earlier using time-dependent perturbation theory and studies the Alfvénic response of field lines to a variety of driving terms; section 4 discusses the revised equations that are appropriate for describing the asymptotic state of the field; finally, section 5 summarises our main results and concludes the paper.

## 2. BASIC EQUATIONS

The coordinate system used throughout this paper is an orthogonal curvilinear one based upon the magnetic geometry. We define three spatial coordinates $(\alpha, \beta, \gamma)$ and let $\hat{\gamma}$ be parallel to the local background magnetic field direction everywhere. The transverse coordinates $(\alpha, \beta)$ are constant on any background line of force and are similar to Euler potentials or Clebsch variables. The background magnetic field is assumed to be solenoidal and irrotational, requiring

$$Bh_\alpha h_\beta = f(\alpha, \beta) \tag{1}$$

$$Bh_\gamma = g(\gamma) \tag{2}$$

where $f$ and $g$ are arbitrary functions of their arguments and the scale factors $h_i$ are equal to $1/|\nabla i|$; $i = \alpha, \beta, \gamma$.

[1] Now at Department of Mathematical and Computational Sciences, University of St Andrews, St Andrews, Scotland



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A physical interpretation of the scale factors may be realized by noting that a real space element $dr$ is equal to $\hat{\alpha} h_\alpha d\alpha + \hat{\beta} h_\beta d\beta + \hat{\gamma} h_\gamma d\gamma$. These are standard properties of such a coordinate system [*Davis and Snider*, 1979]. Similar coordinate systems have facilitated earlier investigations of related problems [*Singer et al.*, 1981; *Southwood and Hughes*, 1983; *Walker*, 1987; *Wright*, 1990; *Wright and Smith*, 1990].

To proceed further we introduce the nonlinear cold ideal MHD equations. For the total velocity and magnetic fields ($\mathbf{U}$ and $\mathbf{B}_T$) and total plasma density $\rho$, the momentum, induction, and continuity equations are

$$\rho\frac{\partial \mathbf{U}}{\partial t} + \rho(\mathbf{U}\cdot\nabla)\mathbf{U} = (\mathbf{B}_T\cdot\nabla)\mathbf{B}_T/\mu_0 - \nabla(B_T^2/2\mu_0) \quad (3)$$

$$\frac{\partial \mathbf{B}_T}{\partial t} = \nabla\wedge(\mathbf{U}\wedge\mathbf{B}_T) \quad (4)$$

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\mathbf{U}) \quad (5)$$

Some of the analysis techniques employed in section 3 are simplest when applied to nondimensional quantities. For this reason we shall assume all quantities are normalized by representative values of the background medium. (For example, lengths can be divided by the equatorial standoff distance of the magnetopause. Magnetic fields, velocity fields, and plasma density can be measured relative to the field strength, Alfvén speed, and density at the nose of the magnetopause.) We shall now look for small disturbances in the dimensionless magnetic and velocity fields ($\mathbf{b}$ and $\mathbf{u}$) about a magnetostatic equilibrium field $\mathbf{B}$ and density distribution $\rho_0$. The perturbations may be written as a series of functions

$$\mathbf{b} = \sum_n \epsilon^n \mathbf{b}^{(n)}; \qquad \mathbf{u} = \sum_n \epsilon^n \mathbf{u}^{(n)} \quad (6)$$

and similarly for the density disturbance. Utilizing standard expressions for grad and curl the momentum and induction equations become to first order in $\epsilon$

$$\frac{\mu_0\rho_0}{B}\cdot\frac{\partial \mathbf{u}}{\partial t} = \frac{\hat{\alpha}}{h_\alpha h_\gamma}\cdot\left[\frac{\partial}{\partial\gamma}(b_\alpha h_\alpha) - \frac{\partial}{\partial\alpha}(b_\gamma h_\gamma)\right] + \frac{\hat{\beta}}{h_\beta h_\gamma}\cdot\left[\frac{\partial}{\partial\gamma}(b_\beta h_\beta) - \frac{\partial}{\partial\beta}(b_\gamma h_\gamma)\right] + \hat{\gamma}0 \quad (7)$$

$$\frac{\partial \mathbf{b}}{\partial t} = \frac{\hat{\alpha}}{h_\beta h_\gamma}\cdot\frac{\partial}{\partial\gamma}(u_\alpha h_\beta B) + \frac{\hat{\beta}}{h_\alpha h_\gamma}\cdot\frac{\partial}{\partial\gamma}(u_\beta h_\alpha B) - \frac{\hat{\gamma}}{h_\alpha h_\beta}\cdot\left[\frac{\partial}{\partial\alpha}(u_\alpha h_\beta B) + \frac{\partial}{\partial\beta}(u_\beta h_\alpha B)\right] \quad (8)$$

The functions $\mathbf{b}$ and $\mathbf{u}$ in these equations are actually $\mathbf{b}^{(1)}$ and $\mathbf{u}^{(1)}$, but we have omitted the cumbersome superscripts. The next section studies the time-dependent solution of the linearized equations (7) and (8).

## 3. TIME-DEPENDENT SOLUTIONS

For definiteness we shall seek the solution of transverse (Alfvén) fields with a $\hat{\beta}$ component. This does not restrict the applicability of our calculations since we have not prescribed the orientation of the coordinates $(\alpha,\beta)$. (If $\hat{\beta}$ were aligned with the azimuthal direction, our results would describe toroidal pulsations; if $\hat{\beta}$ were directed across L shells, we would describe poloidal pulsations.) The $\hat{\beta}$ components of the momentum and induction equations may be combined to yield a driven Alfvén wave equation for the $b_\beta$ field [*Inhester*, 1986],

$$\frac{\partial}{\partial\gamma}\left(\frac{V^2 h_\alpha}{h_\beta h_\gamma}\cdot\frac{\partial}{\partial\gamma}(b_\beta h_\beta)\right) - h_\alpha h_\gamma\cdot\frac{\partial^2 b_\beta}{\partial t^2} = \frac{\partial}{\partial\gamma}\left(\frac{V^2 h_\alpha}{h_\beta h_\gamma}\cdot\frac{\partial}{\partial\beta}(b_\gamma h_\gamma)\right) \equiv C \quad (9)$$

where $V$ is the Alfvén speed. (Note that *Inhester* [1986] employs an alternative definition for his scale factors.)

In addition to the driven Alfvén wave equation (9) there is a fast mode wave equation which is driven by the Alfvén wave fields. The driving terms represent coupling between the fast and Alfvén modes. *Wright* [1992] shows that for small azimuthal wave numbers ($m < 4$), first-order (in $m$) Alfvén fields are excited (according to (9)), whilst there is only a second-order (in $m$) correction to the fast mode. In this approximation we may neglect the effect of the Alfvén waves on the fast mode.

Setting the right-hand side of (9) to zero yields the decoupled Alfvén wave equation for $b_\beta$. Evidently this is a good approximation for toroidal oscillations in an axisymmetric medium ($\partial/\partial\beta = 0$) [*Dungey*, 1967]. Besides axisymmetric systems, the use of the decoupled wave equation imposes constraints upon the geometry of the magnetic field and the distribution of plasma density [*Wright*, 1990; *Wright and Evans*, 1991] which are seldom realized in practice. Nevertheless, the decoupled equations permit us to calculate the normal modes of the system which can yield insights into the likely behavior of complicated media. This approach has proved useful in the past for investigating terrestrial pulsations [*Dungey*, 1967; *Radoski*, 1967; *Cummings et al.*, 1969; *Warner and Orr*, 1979; *Singer et al.*, 1981] and Alfvén waves in Jupiter's magnetosphere [*Glassmeier et al.*, 1989; *Smith and Wright*, 1989; *Wright and Smith*, 1990].

Notice how the decoupled Alfvén wave equation (9) (with $C = 0$) only contains spatial derivatives along a single background line of force, as one would expect from the guided nature of Alfvén wave propagation. *Wright and Smith* [1990] note that this equation is of the Sturm-Liouville form [e.g., *Morse and Feshbach*, 1953]. If the ionosphere is a massive perfect conductor, we may set the electric field in the ionosphere (and immediately above it) to zero. As a result the velocity perturbation must vanish as the ionosphere is approached. Such a boundary condition neglects ionospheric dissipation, and not surprisingly Sturm-Liouville theory tells us that the natural frequencies of the Alfvén waves will be real (i.e., the solutions are not damped). Each eigenfrequency is associated with an eigenmode (or harmonic) of the system. The values of the eigenfrequencies and structure of the eigenmodes have been calculated for realistic models of the terrestrial [*Cummings et al.*, 1969; *Singer et al.*, 1981] and Jovian [*Wright and Smith*, 1990] magnetospheres. The $n$th mode and frequency satisfy the following equation, subject to the boundary condition $\partial(b_\beta h_\beta)/\partial\gamma = 0$ at the ionosphere.

$$\frac{\partial}{\partial\gamma}\left(\frac{V^2 h_\alpha}{h_\beta h_\gamma}\cdot\frac{\partial}{\partial\gamma}(b_{\beta n} h_\beta)\right) + \omega_{\beta n}^2 h_\alpha h_\gamma b_{\beta n} = 0 \quad (10)$$

For every field line $(\alpha, \beta)$ there will be an infinite set of modes and frequencies. Further standard properties include the orthogonality of any two different modes on a given field line,

$$\int_{\gamma_1}^{\gamma_2} b_{\beta i} b_{\beta j} h_\alpha h_\beta h_\gamma d\gamma = \delta_{ij} \tag{11}$$

The limits $\gamma_1(\alpha, \beta)$ and $\gamma_2(\alpha, \beta)$ label the ionospheric ends of the field lines. *Chen and Cowley* [1989] have discussed these eigenfunctions and their orthonormality for a three-dimensional dipolar background field. (Note that the normalization introduced in (11) determines the amplitude of each eigenmode.) Moreover, the set of eigenmodes on any field line forms a complete set in the sense that we may write an arbitrary $b_\beta$ disturbance as a sum over these modes,

$$b_\beta(\alpha_0, \beta_0, \gamma, t) = \sum_n a_{\beta n}(\alpha_0, \beta_0, t) b_{\beta n}(\alpha_0, \beta_0, \gamma) \tag{12}$$

Once again, $(\alpha_0, \beta_0)$ can be thought of as parameters which specify our field line; they will be omitted from the remainder of this section. Each field line has an associated set of coefficients $\{a_{\beta n}; n = 0, 1, 2...\}$ giving the amplitudes of the modes on that particular field line. In general one can think of the coefficients as being continuous functions of $\alpha$ and $\beta$ as well as of time.

Our problem of determining the $b_\beta$ fields has now become the problem of determining the coefficients $\{a_{\beta n}\}$. For the decoupled equation (9), with $C = 0$, it is straightforward: The time dependence of each mode is simply oscillatory at the appropriate eigenfrequency, $a_{\beta n} \propto \exp[i\omega_{\beta n}]$, representing the free oscillation of each harmonic. The amplitude and phase of each mode may be determined from some prescribed initial conditions. This approach has proved useful for studying the evolution of Io's Alfvén waves when it is assumed that coupling to the fast mode is negligible [*Smith and Wright*, 1989; *Wright and Smith*, 1990]. If coupling with the fast mode cannot be neglected, we must solve the inhomogeneous equation (9) with $C \neq 0$, which is a far more complicated problem. Similar equations have been investigated in quantum mechanics via time-dependent perturbation theory, which is the technique we shall employ here.

The method of time-dependent perturbation theory begins by writing the $b_\beta$ field as a sum over the eigenmodes, as is done in (12). The sum may be substituted into the coupled wave equation (9) and will remove the awkward field-aligned derivatives according to the definition of the modes in (10). (Note that the coefficients $\{a_{\beta n}\}$ do not vary along a given field line.) The equation governing the coefficient of any mode may be found by employing the orthonormal relation (11); simply multiply the inhomogeneous equation by the desired harmonic (and $h_\beta$) and integrate along the field line. The result is an infinite set of equations, $n = 1, 2, 3...$,

$$\frac{d^2 a_{\beta n}}{dt^2} + \omega_{\beta n}^2 a_{\beta n} = -\int_{\gamma_1}^{\gamma_2} h_\beta b_{\beta n} C d\gamma \equiv C_n(t) \tag{13}$$

each of which is of the form of a driven undamped harmonic oscillator (i.e., infinite $Q$), the solutions of which are well known [*Morse and Ingard*, 1968]. The effect of nonuniformity along the background field lines has the effect of producing eigenfunctions that are not sinusoidal. *Southwood and Kivelson* [1986] have shown how such an inhomogeneity can lead to enhanced or suppressed excitation of different modes. Our formulation demonstrates this property via the overlap integral of $C$ and the mode in question. The result is to determine the driver $C_n$ in the equation governing $a_{\beta n}$, (13), and consequently how effectively that mode may be excited. Of course, the detailed variation in $C_n$ depends upon the structure of the fast mode, which we do not consider explicitly here.

In systems which are uniform along **B**, the cavity modes and the Alfvén modes share a harmonic dependence along field lines [*Allan et al.*, 1987; *Southwood*, 1974]. This means that only a single Alfvén mode may be excited on each field line for each cavity mode. In the more general case described by (13) we see a single cavity mode may excite, to some degree, all the Alfvén modes on each field line. This effect is somewhat akin to the 'mixing of states' found in quantum theory.

The solutions of (13) are well known for a variety of simple driving functions. However, it is possible to evaluate the response of $a_{\beta n}(t)$ for quite arbitrary drivers in terms of an integral over the Green's function [*Morse and Feshbach*, 1953]. For example, if the field line is initially unperturbed ($a_{\beta n}$ and its derivative are both zero at $t = -\infty$), the solution for $a_{\beta n}(t)$ is

$$a_{\beta n}(t) = \int_{-\infty}^{t} \frac{C_n(t')}{\omega_{\beta n}} \cdot \Big[\cos(\omega_{\beta n} t')\sin(\omega_{\beta n} t) - \sin(\omega_{\beta n} t')\cos(\omega_{\beta n} t)\Big] dt' \tag{14}$$

### *Steady Harmonic Driver*

If we wish to study the excitation of resonant Alfvén waves within the context of a Kelvin-Helmholtz driven system [*Southwood*, 1974; *Chen and Hasegawa*, 1974] or a cavity mode model with weak damping (and $m < 4$) [*Kivelson and Southwood*, 1985], a suitable driving term that would represent the effect of the compressional fast mode would be

$$C_n(t) = 0 \qquad t < 0 \tag{15a}$$

$$C_n(t) = C_{n0}\sin(\omega_c t), \qquad t \geq 0 \tag{15b}$$

where $\omega_c$ is the natural frequency of the compressional/cavity mode. Of course, several different cavity modes may be present at any one time and can be accommodated into our theory by writing $C_n(t)$ as a sum over several oscillatory terms, each term representing the influence of a single cavity mode. For the remainder of this subsection we shall concentrate upon calculating the Alfvén response due to the driving function in (15). If the frequency of the compressional/cavity mode $\omega_c$ does not equal the natural Alfvén frequency $\omega_{\beta n}$, then a driven off-resonance response is found ($\omega_{\beta n} \neq \omega_c$)

$$a_{\beta n} = 0 \qquad t < 0 \tag{16a}$$

$$a_{\beta n} = \frac{C_{n0}}{\omega_c^2 - \omega_{\beta n}^2} \cdot \left[-\sin(\omega_c t) + \frac{\omega_c}{\omega_{\beta n}} \cdot \sin(\omega_{\beta n} t)\right], \quad t \geq 0 \tag{16b}$$

When the driving frequency is much less than the natural frequency ($\omega_c \ll \omega_{\beta n}$), we see that $a_{\beta n} \approx (C_{n0}/\omega_{\beta n}^2)\sin(\omega_c t)$, which displays the expected in-phase oscillation of the driven Alfvén fields and the driving function. If the natural frequency is much less than the driving frequency ($\omega_{\beta n} \ll \omega_c$), then $a_{\beta n} \approx (C_{n0}/\omega_c\omega_{\beta n})\sin(\omega_{\beta n}t)$. Now the principal response is a natural oscillation of the field line (excited by the abrupt change of the driver at $t = 0$). In this limit, note how the much smaller driven component of $a_{\beta n}$ is $\approx -(C_{n0}/\omega_c^2)\sin(\omega_c t)$, which is out of phase with the driver by $\pi$, as one would expect.

The expression given in (16) is valid when ($\omega_{\beta n} \neq \omega_c$), and it describes the nonresonant Alfvén response of background field lines. We see that when a resonance is approached ($\omega_c \to \omega_{\beta n}$) the amplitude of the Alfvén fields can become very large due to the vanishing of the denominator. Indeed, expression (16) appears to be singular in the resonant limit. This is not actually true, and (16) can be manipulated to give the following resonant response. (This result is also found by evaluating (14) when we set $\omega_c = \omega_{\beta n}$.)

$$a_{\beta n} = 0 \qquad t < 0 \quad (17a)$$

$$a_{\beta n} = C_{n0}\left(\frac{\sin(\omega_{\beta n}t)}{2\omega_{\beta n}^2} - \frac{t}{2\omega_{\beta n}}\cdot\cos(\omega_{\beta n}t)\right), t \geq 0 \quad (17b)$$

The most important feature in (17) is the secular term which represents the steady growth in amplitude of the $n$th mode. Indeed, this result demonstrates the time-dependent growth of a resonance in an arbitrary medium and is a generalization of earlier studies [*Inhester*, 1986; *Southwood and Kivelson*, 1986; *Mond et al.*, 1990; *Southwood and Kivelson*, 1990]. Note how the secular term ($-(C_{n0}t/2\omega_{\beta n})\cos(\omega_{\beta n}t)$) lags behind the driver by a phase of $\pi/2$, as one would expect for a resonantly driven oscillator. The amplitude of this secular term grows linearly with time and will dominate the homogenous part of the solution ($(C_{n0}/2\omega_{\beta n}^2)\sin(\omega_{\beta n}t)$) after a fraction of a cycle, $t > 1/\omega_{\beta n}$. It is interesting to note that the rate of increase in amplitude will tend to be greater when $\omega_{\beta n}$ is smaller, suggesting that the lower-frequency modes will grow more rapidly. (A full calculation would have to allow for the fact that $C_{n0}$ would, of course, be different for each Alfvén eigenmode.)

A convenient way to quantify the amplitude of the disturbance over many cycles when $t$ is large is in terms of the root-mean-square amplitude, $< a_{\beta n} >$. We find that

$$< a_{\beta n}(t \to \infty) > \cdot \frac{\omega_c^2}{C_{n0}} = \frac{1}{x(1-x^2)}\cdot\sqrt{\frac{x^2+1}{2}} \qquad x \neq 1 \quad (18)$$

where $x = \omega_{\beta n}/\omega_c$. Figure 1 plots $< a_{\beta n}(t \to \infty) >$ as a function of $\omega_{\beta n}$. If we are interested in studying toroidal magnetic pulsations, the natural variation of $\omega_{\beta n}$ with $L$ shell means we can also regard the horizontal axis in Figure 1 as an $L$ shell coordinate ($\omega_{\beta n}$ increases as $L$ decreases). Thus if $\omega_{\beta n}(L) = \omega_c$ is satisfied on the resonant $L$ shell ($L_r$), then $\omega_{\beta n}(L)/\omega_c > 1$ corresponds to $L$ shells earthward of the resonance ($L < L_r$), and $\omega_{\beta n}(L)/\omega_c < 1$ corresponds to $L$ shells $> L_r$. From the discussion following (16) we would expect that on $L$ shells $< L_r$ the main contribution to $< a_{\beta n} >$ arises from the Alfvén mode oscillating with the cavity mode frequency. In contrast, on $L$ shells $> L_r$ we expect the Alfvén fields to oscillate with the natural Alfvén frequency of that $L$ shell. We return to these predictions in section 5.

The resonant result given in (17) is important because it demonstrates how a magnetic pulsation may become established in a general magnetic field configuration. Evidently we cannot rely upon (17) to describe our fields reliably for an indefinite time, since the amplitude of our perturbed field would become very large and invalidate our initial linearisation (7) and (8). For example, after a time $t \sim \omega_{\beta n}/(\epsilon C_{n0})$

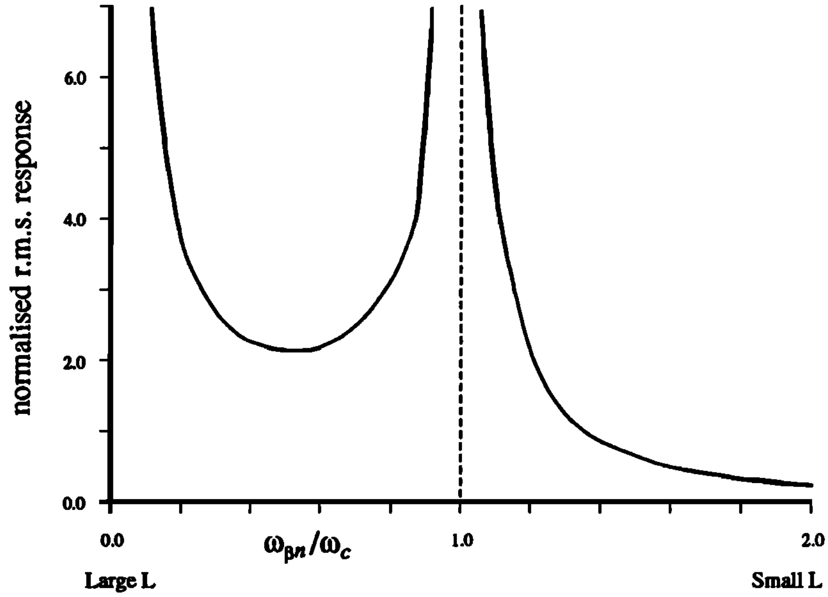


Fig. 1. The variation of $< a_{\beta n} > \omega_c^2/C_{n0}$ as a function of the natural frequency of the mode ($\omega_{\beta n}$) for the steady harmonic driver (15) when $t \gg 1/\omega_{\beta n}, 1/\omega_c$. The vertical dashed line is located at the resonant frequency where the amplitude is singular. Large amplitudes are also found for modes whose natural frequency is much less than the driving frequency. In an axisymmetric magnetospheric model of toroidal pulsations we can think of the horizontal axis as also representing the $L$ shell coordinate. The right-hand end of the axis will correspond to small $L$ near the plasmasphere, while the left-hand end of the axis will correspond to large $L$ near the magnetopause. The predominant contribution to $< a_{\beta n} >$ to the right of the asymptote (earthward of the resonant $L$ shell) is due to oscillations at the driving frequency, whereas to the left of the asymptote, $< a_{\beta n} >$ arises mainly from oscillations at the natural frequency $\omega_{\beta n}$.

the perturbed fields will be of the same order as the background fields, and we shall not be able to neglect nonlinear terms. Problems arising from large perturbed fields can be avoided to some extent by including ionospheric dissipation. In this case the resonance builds up to an amplitude where the rate of energy dissipation in the ionosphere is equal to the rate of resonantly absorbed energy from the cavity mode [*Inhester*, 1986]. The inclusion of dissipation is beyond the scope of the present paper; however, in the next section we do consider the asymptotic state of a resonantly driven pulsation by including nonlinear terms. For the moment we shall focus our attention upon the growth of Alfvén fields when the linearised equations (7) and (8) are valid.

*Pulse Driver*

Cavity modes and the fast mode excited by the Kelvin-Helmholtz instability are not the only suitable mechanism for producing a driving term in the Alfvén wave equation (9). It has been suggested that the motion of reconnected flux along the dayside magnetopause will disturb adjacent closed magnetospheric field lines [*Southwood and Kivelson*, 1990]. Recent observations demonstrate how waves can be excited by the collision of a small dense plasma cloud in the solar wind with the magnetopause [*Lühr et al.*, 1990]. In both these cases, the driving term would be not an oscillatory function but a short pulse of fixed duration. We do not wish to model the driving term in detail for these mechanisms here. Instead we focus upon the types of solutions that are found for the Alfvén modes when a pulse driver of the following form is used.

$$C_n(t) = 0 \qquad t < 0 \qquad (19a)$$

$$C_n(t) = \frac{C_{n0}}{2} \cdot (1 - \cos(2\pi t/\tau_c)) \qquad 0 \le t < \tau_c \qquad (19b)$$

$$C_n(t) = 0 \qquad \tau_c \le t \qquad (19c)$$

The fixed duration of the pulse driver is the time interval $\tau_c$. In a crude sense we can think of the frequency associated with the driver (19) as being equal to $2\pi/\tau_c$, and *Southwood and Kivelson* [1990] have suggested that the largest-amplitude response in parallel current density will occur on field lines with a natural Alfvén frequency that satisfies $\omega_{\beta n} = 2\pi/\tau_c$. The response of the Alfvén fields is easily calculated by evaluating (14). For normal modes whose natural frequency does not coincide with the associated frequency of the driver ($\omega_{\beta n}\tau_c \ne 2\pi$) we find an amplitude of

$$a_{\beta n} = 0 \qquad t < 0 \qquad (20a)$$

$$a_{\beta n} = \frac{C_{n0}}{2} \cdot \left[\frac{1}{\omega_{\beta n}^2} + \frac{\cos(\omega_{\beta n} t)}{\omega_{\beta n}^2(\omega_{\beta n}^2\tau_c^2/4\pi^2 - 1)} - \frac{\cos(2\pi t/\tau_c)}{\omega_{\beta n}^2 - 4\pi^2/\tau_c^2}\right], \qquad 0 \le t < \tau_c \qquad (20b)$$

$$a_{\beta n} = \frac{-C_{n0}}{\omega_{\beta n}^2(\omega_{\beta n}^2\tau_c^2/4\pi^2 - 1)} \cdot \sin(\omega_{\beta n} t - \omega_{\beta n}\tau_c/2) \times \sin(\omega_{\beta n}\tau_c/2), \qquad \tau_c \le t \qquad (20c)$$

The long-term behavior of the Alfvén fields ($t \ge \tau_c$) is just a free oscillation of the undriven eigenmode. It is interesting to note that the magnitude of the driven fields when $\tau_c\omega_{\beta n} \ll 1$ (i.e., the frequency of the normal mode is much smaller than the associated frequency of the driver) scales as $C_{n0}/\omega_{\beta n}^2$. In this limit there will be a tendency for the lower-frequency modes to achieve a larger amplitude. (A full calculation must allow for the fact that $C_{n0}$ will be different for each mode.) In the other extreme, when the frequency of the driver is much less than the frequency of the normal mode being considered ($\tau_c\omega_{\beta n} \gg 1$), we find that the magnitude of the disturbance scales according to $4\pi^2 C_{n0}/(\omega_{\beta n}^4\tau_c^2)$, suggesting that modes with a natural frequency greater than the frequency associated with the driver are not excited very effectively. It is also interesting to note the envelope in the amplitude response introduced by the second sine term in the final expression of equation (20): When an integer number (greater than 1) of natural oscillations is equal to the interval $\tau_c$, the driven response is identically zero for that mode, whereas the envelope of $a_{\beta n}$ tends to have local maxima on field lines where the number of natural oscillations of a given mode during the interval $\tau_c$ is an integer plus a half.

The behavior of $a_{\beta n}$ is given in (20) for modes whose natural frequency is different from the frequency associated with the driving term ($2\pi/\tau_c$). Indeed, when this assumption is not satisfied, the expressions for $a_{\beta n}$ appear to be singular. Once again, we find that the solution is not actually singular and can be manipulated (or (14) recalculated with $\omega_{\beta n}\tau_c = 2\pi$) to yield the following evolution for $a_{\beta n}(t)$,

$$a_{\beta n} = 0 \qquad t < 0 \qquad (21a)$$

$$a_{\beta n} = \frac{C_{n0}}{2\omega_{\beta n}^2} \cdot \left[1 - \cos(\omega_{\beta n} t) - \frac{\omega_{\beta n} t}{2} \cdot \sin(\omega_{\beta n} t)\right], \qquad 0 \le t < \tau_c \qquad (21b)$$

$$a_{\beta n} = \frac{-\pi C_{n0}}{2\omega_{\beta n}^2} \cdot \sin(\omega_{\beta n} t) \qquad \tau_c \le t \qquad (21c)$$

The long-term behavior of the Alfvén fields (when $t \ge \tau_c$) is again a free oscillation of a normal mode, as one would expect when the driver is zero. Once again, we find that lower-frequency modes are more readily excited.

In contrast to *Southwood and Kivelson's* [1990] conjecture that the principal response is found as 'resonant' Alfvén modes ($\omega_{\beta n} = 2\pi/\tau_c$), we expect $a_{\beta n}$ to be greatest for the lower harmonic modes. Our results can be clarified by plotting the variation of the r.m.s. value of $a_{\beta n}$ as a function of $\omega_{\beta n}$ following the driven phase, $t > \tau_c$ (see Figure 2). The explicit form of $< a_{\beta n} >$ is

$$< a_{\beta n}(t > \tau_c) > \cdot \frac{4\pi^2}{C_{n0}\tau_c^2} = \frac{\sin(\pi y)}{\sqrt{2}} \cdot \frac{1}{y^2(y^2 - 1)} \qquad y \ne 1 \qquad (22a)$$

$$< a_{\beta n}(t > \tau_c) > \cdot \frac{4\pi^2}{C_{n0}\tau_c^2} = \frac{\pi}{2\sqrt{2}} \qquad y = 1 \qquad (22b)$$

where $y = \omega_{\beta n}\tau_c/2\pi$.

When comparing our results with those of *Southwood and Kivelson* [1990] it should be borne in mind that our calculation concentrates upon the magnetic field perturbation, whereas they consider the parallel current density (their Fig-

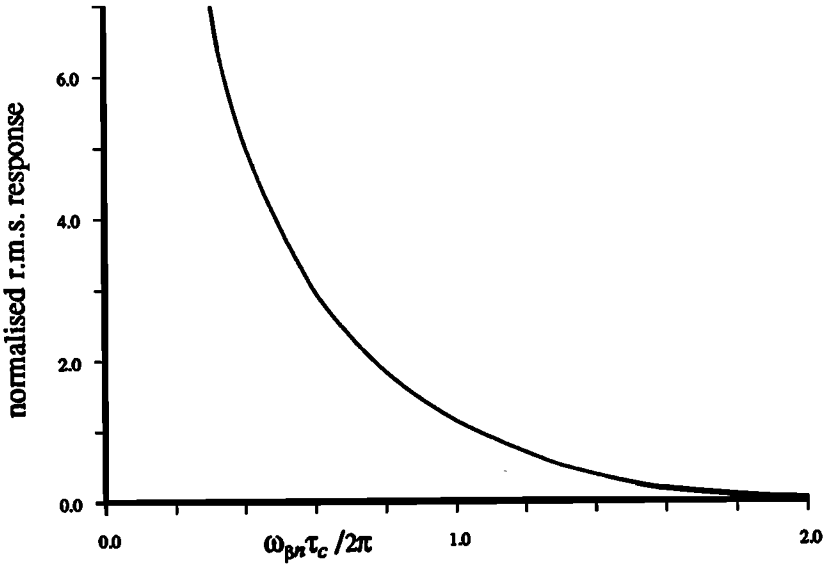


Fig. 2. The variation of $< a_{\beta n} > 4\pi^2/C_{n0}\tau_c^2$ as a function of the natural frequency of the mode ($\omega_{\beta n}$) for the pulse driver (19) when $t > 2\pi/\tau_c$. The Alfvén modes that are excited most effectively have the lowest frequencies, and these is no evidence of any 'resonant' excitation.

ure 7). We have not explicitly calculated the current density, and so cannot compare our results rigorously with those of *Southwood and Kivelson* [1990].

Finally we note that the expression for $< a_{\beta n}(t > \tau_c) >$ given above is equal to the Fourier transform of (19) divided by $\omega_{\beta n}$. This reinforces our conclusion that the low-frequency modes achieve larger amplitudes than the higher-frequency modes.

*Finite Cycle Harmonic Driver*

Evidently the response of the Alfvén fields to the steady harmonic driver (Figure 1) and that to the pulse driver (Figure 2) are very different in character. In practice a realistic driver will be somewhere between the two extremes considered above. For this reason we consider one more type of driving term which is oscillatory in nature, but only for a specified number of cycles. We define the following form for the driver:

$$C_n(t) = 0 \qquad t < 0 \qquad (23a)$$

$$C_n(t) = C_{n0}\sin(\omega_c t) \qquad 0 \le t < \ell\pi/\omega_c \qquad (23b)$$

$$C_n(t) = 0 \qquad \ell\pi/\omega_c \le t \qquad (23c)$$

The above driver is sero except during the time interval $[0, \ell\pi/\omega_c]$ when it executes $\ell$ half cycles at a frequency $\omega_c$ ($\ell = 1, 2, 3...$). The integer $\ell$ will be left as a free parameter in our equations, so that we can study the transition from a simple half cycle, $\ell = 1$ (qualitatively similar to the pulse in (19)) to a steady driver like that in (15), $\ell \to \infty$.

During the time interval $[-\infty, \ell\pi/\omega_c]$ the driver is identical to that given in (15), and the response of $a_{\beta n}$ will be that given in (16) or (17). However, for $t > \ell\pi/\omega_c$ the driver changes from that given in (15), and we must recalculate $a_{\beta n}(t)$ for these times. Performing this calculation we find, for $t > \ell\pi/\omega_c$

$$a_{\beta n} = \frac{C_{n0}}{\omega_c^2 - \omega_{\beta n}^2}\cdot\frac{\omega_c}{\omega_{\beta n}}\left[\left(1 - (-1)^\ell\cos\phi_\ell\right)\sin(\omega_{\beta n}t)\right.$$

$$\left. + (-1)^\ell \sin\phi_\ell\cos(\omega_{\beta n}t)\right] \qquad \omega_{\beta n} \ne \omega_c \qquad (24a)$$

$$a_{\beta n} = \frac{-C_{n0}\ell\pi}{2\omega_c^2}\cdot\cos(\omega_{\beta n}t) \qquad \omega_{\beta n} = \omega_c \qquad (24b)$$

where $\phi_\ell = \ell\pi\omega_{\beta n}/\omega_c$. Of course, for $t > \ell\pi/\omega_c$ the driver is sero and so the Alfvén modes simply oscillate at their natural frequencies. Note how the amplitude of these free oscillations on the resonantly excited field line ($\omega_{\beta n} = \omega_c$) is directly proportional to $\ell$, suggesting that the more cycles the field line is driven over, the larger the final amplitude. The magnitude of the free oscillation following excitation can be expressed most simply in terms of the r.m.s. amplitude $< a_{\beta n} >$ (calculated by integrating the square of (24) over one period, dividing by that period, and finally taking the positive square root). For $t > \ell\pi/\omega_c$ we find

$$< a_{\beta n} > \cdot\frac{\omega_c^2}{C_{n0}} = \frac{1}{x(1 - x^2)}\cdot\sqrt{1 - (-1)^\ell\cos\phi_\ell} \qquad \omega_{\beta n} \ne \omega_c \qquad (25a)$$

$$< a_{\beta n} > \cdot\frac{\omega_c^2}{C_{n0}} = \frac{\ell\pi}{2\sqrt{2}} \qquad \omega_{\beta n} = \omega_c \qquad (25b)$$

where once again $x = \omega_{\beta n}/\omega_c$. Figure 3a plots the variation of the normalised r.m.s. amplitude (i.e., $< a_{\beta n} > \omega_c^2/C_{n0}$) as a function of $\omega_{\beta n}$ following the driven phase for several values of $\ell$. The curve corresponding to $\ell = 1$ represents the response to a half cycle of the driver and is qualitatively similar to the pulse driver (19). Indeed the dependence of the r.m.s amplitude on $\omega_{\beta n}$ is also very similar to that found for the pulse driver (see Figure 2). No resonant behavior is evident from the driver (23) when $\ell = 1$.

Figure 3a also exhibits a lack of resonant behavior when a complete cycle of the driver is executed ($\ell = 2$), and we find

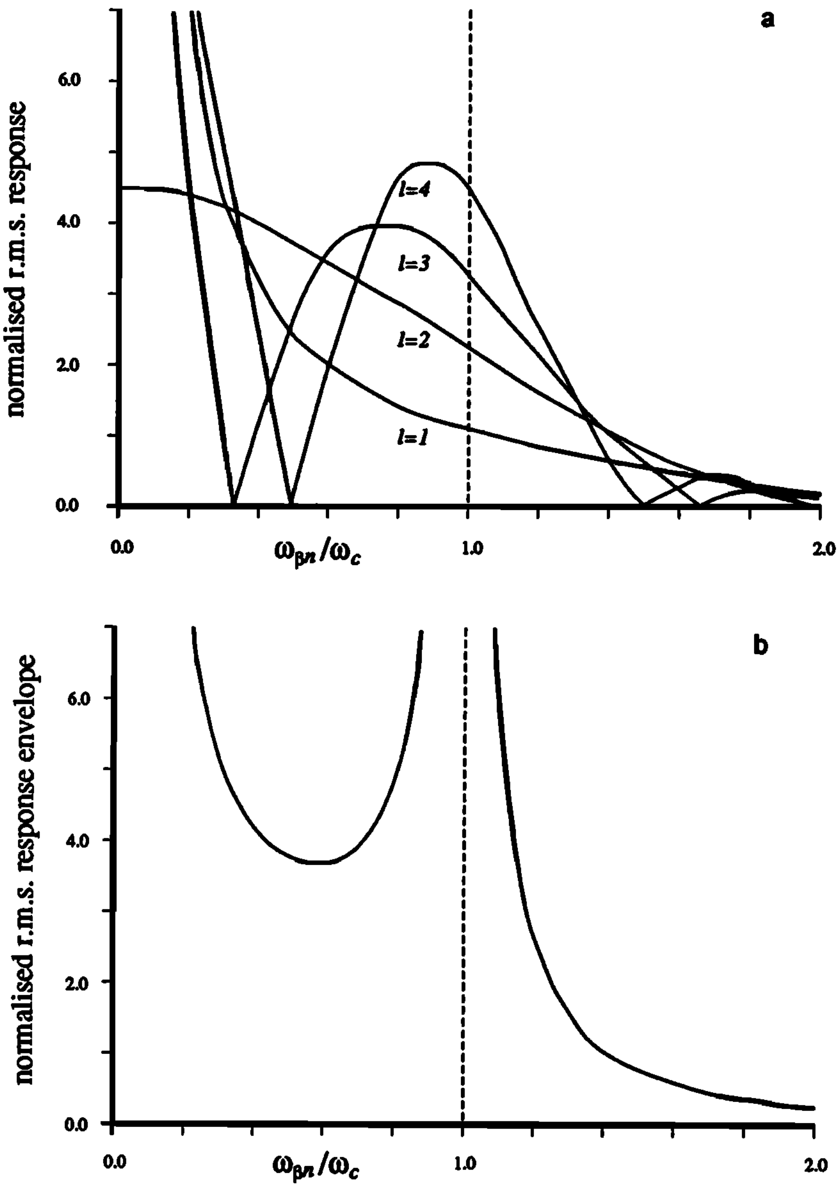


Fig. 3 (a). The variation of $< a_{\beta n} > \omega_c^2/C_{n0}$ as a function of the natural frequency of the mode ($\omega_{\beta n}$) for the driver (23) when $t > \ell\pi/\omega_c$. The four curves illustrate the dependence of the excited Alfvén fields on the number of cycles ($\ell/2$) over which the driver acts. The transition from a pulse-like driver ($\ell = 1$) response to resonant excitation ($\ell \gg 1$) is evident: As the number of driving cycles increases, the peak in the plot moves closer to the resonant asymptote and becomes sharper. This property is shown more clearly in (b) which plots the envelope of a curve like those in Figure 3a for which $\ell \to \infty$.

that lower frequency modes realise larger amplitudes. When the field line is driven for a cycle and a half ($\ell = 3$), a clear peak emerges along with zeros in $< a_{\beta n} >$. However, the peak does not coincide with the 'resonantly excited' field line, but is found for lines where $\omega_{\beta n}$ is slightly less than $\omega_c$. The final curve in Figure 3a is plotted for two complete cycles of the driver (23), $\ell = 4$. The peak is now higher than before and closer to the resonant position. In fact, as $\ell$ increases further, the maximum value of $< a_{\beta n} >$ increases and is located closer to the dashed vertical line in Figure 3a where we expect resonance to occur. This property can be seen by considering the limit $\ell \to \infty$: $< a_{\beta n}(t > \ell\pi/\omega_c) >$ becomes a rapidly varying function of $\omega_{\beta n}/\omega_c$ due to the dependence on the phase $\phi_\ell = \ell\pi\omega_{\beta n}/\omega_c$. However, we can plot the envelope of this function instead, as in Figure 3b.

Note the strong similarity between Figure 1 and Figure 3b confirming the expectation that as $\ell \to \infty$ we recover a similar resonant behavior to that found earlier. Evidently, for small $\ell$ we do not find the largest response on the 'resonant' field lines. The driver must be coherent for at least two cycles to produce a large Alfvén response near the 'resonant' field lines, and even then the adjacent field lines also experience similar excitation. We may quantify how localised the peak in $< a_{\beta n} >$ is in terms of the full width of the normalised envelope $\sqrt{2}/[x(1-x^2)]$ at half the height of the r.m.s. resonance value (equal to $\ell\pi/4\sqrt{2}$). If the full width of the peak is $\Delta\omega_{\beta n}$, it is convenient to define the parameter

$$\Delta_n = \frac{\Delta\omega_{\beta n}}{2\omega_c} \tag{26}$$

in which case the height of the envelope at half maximum is $\sqrt{2}/[2\Delta_n + 3\Delta_n^2 + \Delta_n^3]$. Equating this expression with half of the peak r.m.s. value we can determine $\Delta_n$,

$$\Delta_n = \frac{\Delta\omega_{\beta n}}{2\omega_c} \approx \frac{1}{2}\left(\sqrt{4/9 + 32/(3\ell\pi)} - \frac{2}{3}\right) \qquad \Delta_n < 1 \tag{27a}$$

$$\Delta_n = \frac{\Delta\omega_{\beta n}}{2\omega_c} \approx \frac{4}{\ell\pi} \qquad \Delta_n \ll 1 \tag{27b}$$

(Expression (27a) neglects cubic terms in $\Delta_n$, while (27b) neglects squares also.) A similar estimate for the width of the resonant peak as a function of time (rather than $\ell$) can be derived from (16) and (17) if the driver is continuous (15).

The result (27) demonstrates clearly that the more cycles ($\ell/2$) a field line is driven for, the more localised the principal disturbance is in frequency. In the case of toroidal magnetic pulsations in an axisymmetric magnetosphere, this also means that as $\ell$ increases, the pulsation will be localised to a smaller range of $L$ shells.

## 4. Asymptotic Solutions

The previous section demonstrated, to lowest order, how compressional wave fields due to a variety of sources can couple to transverse Alfvén waves on closed field lines. Of course, the precise form of the coupling coefficients $C_n$ will depend upon the structure of the $b_\gamma$ wave field, which we do not consider in detail here. (See *Wright* [1992] for more details.) When there is significant Alfvén wave excitation, the main response is confined near resonant field lines. If the natural Alfvén frequencies vary across the background field lines, it will often be the case that the excited Alfvén waves are confined to a thin layer of magnetic flux [*Inhester*, 1986]. In this section we consider what the long-term state of the Alfvén fields will be. This explains why toroidal pulsations are highly localised in the direction across magnetospheric $L$ shells. Previous numerical and analytical studies also exhibit singular behavior on resonant field lines [*Southwood*, 1974; *Chen and Hasegawa*, 1974; *Inhester*, 1986; *Inhester*, 1987; *Zhu and Kivelson*, 1988; *Mond et al.*, 1990; *Lee and Lysak*, 1990]. Some early calculations by *Radoski* [1974] demonstrated how the asymptotic, or long-term, state of the disturbance would ultimately be composed of standing Alfvén waves - all of the energy in the fast cavity mode having been expended in exciting the Alfvén waves. *Zhu and Kivelson* [1988] show how the rate of energy absorption by resonant Alfvén waves damps the cavity mode. However, the time scale of the damping is typically 2 orders of magnitude greater than the Alfvén period, and it seems likely that leakage of the cavity mode down the geomagnetic tail will drain energy from the fast mode more quickly than the excitation of a pulsation. Nevertheless, a final state will result in which we have only a thin layer of oscillatory Alfvén fields and no significant cavity mode, as envisaged by *Radoski* [1974].

We may model the asymptotic state as a sheet of field lines containing (to lowest order) perturbed magnetic and velocity fields in the $\hat{\beta}$ direction and being highly localised across the sheet (in $\alpha$). It is a natural feature of such a state that the perturbed fields (at any instant in time) will vary in phase along the resonant sheet. For example, most models impose a dependence $\exp[ik_\beta\beta]$ on the perturbations. The presence of a variation along the sheet is essential if we are to excite a $b_\beta$ Alfvén response. (If $\partial/\partial\beta = 0$, the driving term in (9) is zero.) Given that there must be a phase variation along the sheet it would appear, at first sight, that the Alfvén fields do not satisfy $\nabla \cdot \mathbf{b} = 0$ [*Cross*, 1988; *Rajaram and Venkatesan*, 1990; *Wright*, 1990]. Evidently the solution to the paradox is to include some small $b_\alpha$ fields across the resonant sheet. Moreover, since the scale of variation across the sheet is much smaller than the scale along it in $\beta$, the $\hat{\alpha}$ component of the magnetic field may be much smaller than the main $\hat{\beta}$ component described in section 3.

Let us define the asymptotic state more formally: The dominant fields are resonantly excited $(b_\beta, u_\beta)$ disturbances. These fields vary in phase along the resonant sheet, which we shall assume has a width (in $\hat{\alpha}$) of $\delta$. There are also perturbed fields across the sheet, but these are much smaller than the resonant components, $(b_\alpha, u_\alpha) \ll (b_\beta, u_\beta)$. The disturbances parallel to the background field are in turn much smaller than the $\hat{\alpha}$ field components. The coupling between the two transverse fields has been discussed recently by *Rajaram and Venkatesan* [1990]. They advocate that the small spatial scale $\delta$ can result in significant Hall currents. Whilst the Hall current may be an important feature of some pulsations, we present a complementary solution here within the ideal Ohm's law approximation.

It can be argued on physical grounds that the wave solution should be incompressible to lowest order: Any compressional field perturbation will communicate across a sheet of width $\delta$ on a very small time scale ($\sim \delta/V$) and inhibit plasma compression in planes perpendicular to $\mathbf{B}$. Moreover, since the magnetic pressure gradient in the $\hat{\alpha}$ direction ($\sim Bb_\gamma/\mu_0\delta$) must be a perturbation (i.e., much less than $B^2/\mu_0 L$; $L$ is the scale of the background field), we conclude that $b_\gamma \ll B\delta/L$. Thus $b_\gamma$ (or $\nabla \cdot \mathbf{u}_\perp$) is likely to be zero to lowest order, but may be a second- or third-order perturbation.

The (normalised) width of the resonant sheet $\delta$ plays a central role in determining the character of the fields in our asymptotic solution. We shall assume that $\delta$ is so small that we can neglect the variation of the background field on this scale, and employ techniques from boundary layer theory [*Bender and Orszag*, 1978]. In effect, the parameter $\delta$ is used as a second expansion parameter (the other being $\epsilon$) with which to expand the nonlinear MHD equations (7) and (8). We shall seek a 'distinguished' solution satisfying the criteria given above. The simplest distinguished solution is when $\delta = \epsilon$. It can be seen that the solution furnished by this choice does not meet our criteria: The perturbation magnetic pressure gradient in the $\hat{\alpha}$ direction is $\hat{\alpha}\cdot\nabla(b_\beta^2/2\mu_0)$ and will be of order $\epsilon$ (if $b_\beta \sim \epsilon$). Consideration of the $\hat{\alpha}$ component of the momentum equation tells us that we shall drive $u_\alpha$ perturbations that are also of order $\epsilon$, and consequently produce $b_\alpha$ fields of order $\epsilon$ too (see the $\hat{\alpha}$ component of the induction equation (8)). Obviously the solution found when $\delta = \epsilon$ does not satisfy our requirement that $(b_\alpha, u_\alpha) \ll (b_\beta, u_\beta)$ and is not the appropriate relation between $\delta$ and $\epsilon$.

An alternative distinguished solution is found when $\delta = \sqrt{\epsilon}$. In this case (if $(b_\beta, u_\beta)$ are of order $\epsilon$) the magnetic pressure gradient in $\hat{\alpha}$ is of order $\epsilon^{3/2}$, suggesting that the size of $(b_\alpha, u_\alpha)$ will be of order $\epsilon^{3/2}$ too. So far the new choice of $\delta$ and $\epsilon$ produces the correct ordering between the $\hat{\beta}$ and $\hat{\alpha}$ components of the magnetic and velocity fields. However, it does introduce half-powers of $\epsilon$, and we will have to include these half powers in the expansions (6). A full examination of the nonlinear equations expanded in $\delta$ and $\epsilon$ when $\delta = \sqrt{\epsilon}$ can be shown to yield a consistent solution in which the following are the leading terms: $b_\alpha^{(3/2)}, u_\alpha^{(3/2)}, b_\beta^{(1)}, u_\beta^{(1)}, b_\gamma^{(2)}, u_\gamma^{(2)}, \rho^{(1)}$. The lowest-order contributions to the three components of the induction equation are

$$\frac{\partial b_\alpha^{(3/2)}}{\partial t} = \frac{1}{h_\beta h_\gamma} \cdot \frac{\partial}{\partial \gamma}\left(u_\alpha^{(3/2)} h_\beta B\right) \tag{28a}$$

$$\frac{\partial b_\beta^{(1)}}{\partial t} = \frac{1}{h_\alpha h_\gamma} \cdot \frac{\partial}{\partial \gamma}\left(u_\beta^{(1)} h_\alpha B\right) \tag{28b}$$

$$0 = \frac{\partial}{\partial \alpha}\left(u_\alpha^{(3/2)} h_\beta B\right) + \frac{\partial}{\partial \beta}\left(u_\beta^{(1)} h_\alpha B\right) \tag{28c}$$

The two transverse components of the momentum equation yield the following lowest-order relations,

$$\mu_0 \rho_0 \frac{\partial u_\alpha^{(3/2)}}{\partial t} = \frac{B}{h_\alpha h_\gamma} \cdot \left[\frac{\partial}{\partial \gamma}\left(b_\alpha^{(3/2)} h_\alpha\right) - \frac{\partial}{\partial \alpha}\left(b_\gamma^{(2)} h_\gamma\right)\right] - \frac{b_\beta^{(1)}}{h_\alpha h_\beta} \cdot \frac{\partial}{\partial \alpha}\left(b_\beta^{(1)} h_\beta\right) \tag{29a}$$

$$\mu_0 \rho_0 \frac{\partial u_\beta^{(1)}}{\partial t} = \frac{B}{h_\beta h_\gamma} \cdot \frac{\partial}{\partial \gamma}\left(b_\beta^{(1)} h_\beta\right) \tag{29b}$$

(The lowest-order terms in the parallel component of the momentum equation are of order $\epsilon^2$, while those in the continuity equation are of order $\epsilon$.) Note the importance of nonlinear terms in (29a).

It is evident from the above equations that the $\hat{\beta}$ Alfvén wave solution does indeed decouple; (28b) and (29b) completely determine $b_\beta$. In fact, this solution is a generalisation of the highly asymmetric decoupled poloidal mode found in axisymmetric magnetospheres, first derived by *Dungey* [1954, 1967]. The system of equation fits together in a rather unusual fashion: Once the Alfvén fields $(b_\beta, u_\beta)$ have been determined, they act as a driver for the other wave fields including the magnetic pressure. Thus the asymptotic situation is the complete reverse of the early history of the Alfvén fields described in section 3 where the magnetic pressure acted as a driver for the Alfvén wave fields.

Knowledge of the $(b_\beta, u_\beta)$ fields enables us to calculate the smaller $u_\alpha$ velocity from (28c) required to prevent the plasma from becoming compressed and evolving a magnetic pressure perturbation of order $\epsilon$. The evolution of $b_\alpha$ follows directly from the $u_\alpha$ field via (28a). The compressional magnetic field perturbation ($\epsilon^2$ to lowest order) is determined by (29a), namely that it be whatever is necessary to produce the required $u_\alpha$, given $b_\alpha$ and $b_\beta$. We could go on to discuss the evolution of the $u_\gamma$ velocity and the density perturbation, but these quantities do not affect the lowest-order transverse fields that would be observed in data, so we shall curtail the perturbations here.

The discussion above demonstrates the existence of an Alfvén wave solution for a special thickness of the resonant sheet, $\delta = \sqrt{\epsilon}$. We should consider whether the system is likely to develop into a sheet of this thickness. After all, from equation (27) we anticipate that the width of the resonant sheet will be inversely proportional to the duration of the driver. Thus prolonged excitation could result in a narrower resonant sheet than desired by the distinguished solution $\delta = \sqrt{\epsilon}$. For example, we saw that when $\delta = \epsilon$ there exist $b_\alpha$, $b_\gamma$, and $u_\alpha$ fields of order $\epsilon$. These fields are associated with a fast mode which will transport energy away from the resonant layer. Evidently the system is undergoing some adjustment to prevent the resonant Alfvén fields becoming confined to too thin a layer. A detailed account of evolution of the resonant layer is beyond the scope of the present paper; however, the qualitative behavior may be as follows: The transport of energy away from the resonant sheet will permit the excitation of adjacent field line resonances, and consequent absorption of any fast mode energy radiated by the resonant sheet. The net effect could be to broaden the resonant width $\delta$ until the system approaches the distinguished limit $\delta = \sqrt{\epsilon}$.

It is interesting to note that in an axisymmetric magnetosphere the asymptotic toroidal Alfvén modes ($\beta$ = azimuthal coordinate) can be localised to a L shell on which all field lines share identical natural frequencies. However, for asymptotic poloidal Alfvén modes of large azimuthal wave number ($\beta$ = L shell coordinate) the natural frequencies will vary across the meridian plane. This poses no problem to the solution described here, and it simply requires that the perturbation fields $(u_\alpha, b_\alpha, b_\gamma)$ vary accordingly in the meridian plane. In fact we could generalise our solution further by allowing our principal $(b_\beta, u_\beta)$ perturbations to be confined to a surface $\alpha$ =const. This surface may have an arbitrary form throughout space; provided that the scale on which the perturbations vary across the surface ($\delta$) is of order $\sqrt{\epsilon}$, our expansions (28) and (29) remain valid.

## 5. Discussion and Conclusions

In this paper we have studied how magnetic pulsations may become established in arbitrary magnetoplasmas (which carry no background current) and have also considered the long-term solution of any Alfvén fields that are excited. The former calculation takes advantage of various properties of the normal modes of the system. The problem is then rephrased in terms of calculating how the coefficients of each mode evolve in time. Under the simplifying assumption of perfectly reflecting ionospheres it is found that each coefficient is governed by a driven harmonic oscillator equation of infinite $Q$. The driving term for any mode is an overlap integral (along the field line) of the mode in question with the magnetic pressure gradient. Of course, the growth of transverse field and flow perturbations will in turn generate plasma compression and affect the magnetic pressure driving term. However, if the scale on which the perturbations vary along the perturbed surface (say $2\pi/k_\beta$) is much greater than the width of the perturbed surface ($\delta$), then we may neglect the plasma compression due to $u_\beta$ motions. (Such an ordering of lengths will be a natural feature of a resonant response and is borne out by numerical and analytical calculations [*Southwood*, 1974; *Chen and Hasegawa*, 1974; *Inhester*, 1987; *Allan et al.*, 1986a; *Zhu and Kivelson*, 1988; *Lee and Lysak*, 1990].) Thus we are able to calculate the coefficient of any mode as a function of time and so construct the complete Alfvénic disturbance.

The driving term in the equation governing the evolution of each coefficient demonstrates how the variation of the Alfvén mode and the fast mode along the field line may influence how effectively a given mode is excited [*Southwood and*

*Kivelson*, 1986]. For example, if one is an odd function and the other is even, there will be no excitation. In simplified models in which background quantities do not vary along the field lines, fast and Alfvén modes both share a harmonic dependence upon the field-aligned coordinate [*Southwood*, 1974; *Allan et al.*, 1986a]. In this situation the fundamental cavity mode may only excite a fundamental Alfvén response. However, in an inhomogeneous medium where the modes do not have a sinusoidal field-aligned variation, the fundamental cavity mode may excite all the harmonics of the Alfvén modes [*Lee and Lysak*, 1989].

The amplitude of the Alfvén response has been calculated in detail for three types of driving term. The first driving term was a steady oscillatory source (15). On resonant field lines the amplitude of the resonant Alfvén mode grows steadily in time, and lags behind the driving term by a phase of $\pi/2$. The rate of growth of the resonant mode tends to be greater the lower the resonant frequency. On either side of the resonance large-amplitude fields may be produced, and the phase of these changes by $\pi$ across the resonance (relative to the driver) in agreement with other calculations [*Zhu and Kivelson*, 1988] (see Figure 1). We also found that modes $b_{\beta n}$ with $\omega_{\beta n} > \omega_c$ tend to oscillate at the driving frequency and are not strongly excited, whereas normal modes for which $\omega_{\beta n} < \omega_c$ achieve larger amplitudes and tend to oscillate at their natural frequency.

Recent calculations by *Lee and Lysak* [1989] have investigated the excitation of toroidal magnetic fields in a dipole magnetic geometry for low azimuthal wave number ($m = 3$). Many of the features found in their results can be understood within the framework of the theory presented in section 3, and we shall discuss these here briefly. The model employed by *Lee and Lysak* [1989] has a realistic variation in Alfvén speed such that the natural frequency of the toroidal Alfvén modes is a function of $L$ shell. In fact the frequency of any given mode increases as one moves closer to Earth (i.e., smaller $L$). To compare our predictions with the results of *Lee and Lysak* we should align our $\hat{\beta}$ vector with the azimuthal direction. (The plates we refer to in this section are those of *Lee and Lysak* [1989] and may be found at the back of the December 1989 issue of the *Journal of Geophysical Research*).

The frequency spectrum of the cavity modes excited by an impulse at the magnetopause is shown in Plate 1a of *Lee and Lysak* [1989]. Any single cavity mode tends to extend throughout the entire magnetosphere, but may have a complicated nodal structure. Within our model, these cavity modes will be taken as oscillatory drivers for the toroidal Alfvén wave equations. Plate 1b illustrates the frequency spectrum of the toroidal fields excited by the cavity modes as a function of $L$ shell. The trajectories of the natural Alfvén frequencies as a function of $L$ are very clear (for the fundamental, 3nd, 5th, 7th, 9th and 11th harmonics; the initial impulse does not excite the even harmonics). In *Lee and Lysak's* axisymmetric model the natural frequency of a given harmonic is a function of $L$ alone and decreases as $L$ increases. Their Plate 1b demonstrates that on $L$ shells where one of the natural Alfvén frequencies coincides with one of the cavity mode frequencies, large-amplitude toroidal fields are produced; i.e., there is resonance, as anticipated by our equation (17).

The largest resonance is a third harmonic ($f = 0.027$ Hz) at $L = 7.4$ and is located where the resonant cavity mode has a significant amplitude, i.e., $C_3$ will be large. It is interesting to note that smaller-amplitude resonances will occur on $L$ shells where the cavity mode may have a small amplitude, such as near nodes. (See, for example, the 5th toroidal harmonic at $L \approx 5.9$ ($f = 0.090$ Hz) which is excited less effectively than the same harmonic at $L \approx 5.7$ ($f = 0.097$ Hz).)

Besides the resonant excitation of toroidal fields, we see evidence of nonresonant coupling ($\omega_{\beta n}(L) \neq \omega_c$) as anticipated by our equation (16). For example, when the natural Alfvén frequency is less than the cavity mode frequency ($\omega_{\beta n}(L) < \omega_c$) we expect the response to be dominated by an oscillation of the Alfvén mode at its natural frequency. For example, Plate 1b illustrates significant fundamental mode excitation from $L \approx 5.5$ to 9.0 despite the lowest cavity mode frequency being greater than the fundamental toroidal frequency in this interval.

The other nonresonant excitation predicted by equation (16) is the driven response, when the toroidal modes oscillate with the same frequency as the cavity mode, not their natural frequency. We expect this behavior to dominate when $\omega_{\beta n}(L) > \omega_c$. Thus on moving away from a resonance ($\omega_{\beta n}(L) = \omega_c$) in an earthward direction ($\omega_{\beta n}(L) > \omega_c$) we should see toroidal fields oscillating with the cavity mode frequency. This explains the toroidal fields found in Plate 1b earthward of the 3rd harmonic resonances at $L \approx 7.4$ ($f = 0.027$ Hz) and $L \approx 5.6$ ($f = 0.060$ Hz), and the 5th harmonic resonance at $L \approx 6.7$ ($f = 0.060$ Hz). If, on the other hand, we move from a resonance ($\omega_{\beta n}(L) = \omega_c$) to greater $L$ (i.e., $\omega_{\beta n}(L) < \omega_c$) we expect the natural oscillation of toroidal fields to dominate the driven component. This explains the absence of toroidal fields oscillating at the cavity mode frequency on L shells greater than the resonant L shell. Thus we would predict that the toroidal fields found between $L \approx 5.6$ and 7.0 oscillating at the cavity mode frequency ($f = 0.027$ Hz) are actually 3rd harmonics of the field lines, not fundamentals. (Indeed, it is just possible to resolve some 3rd harmonic structure in Plate 3b, which has been filtered to display fields that oscillate with the lowest-frequency cavity mode, $f = 0.027$ Hz.)

The second type of driving term investigated in section 3 was a pulse of fixed duration (19). The resulting coefficients of the Alfvén fields are calculated in terms of the duration of the forcing driver $\tau_c$ and the natural frequencies. When the natural period of the wave is equal to $\tau_c$, secular growth is observed during the driven phase, and the 'resonant' coefficient achieves an amplitude of $C_{n0}\tau_c^2/8\pi$. The coefficients of modes whose natural periods are much smaller than $\tau_c$ (i.e., $\tau_c\omega_{\beta n} \gg 1$) are not excited very effectively. A surprising result is found for modes whose natural period is much greater than $\tau_c$ (i.e., $\tau_c\omega_{\beta n} \ll 1$): The coefficients of these modes are approximately $C_{n0}/\omega_{\beta n}^2$ which means that the lower the natural frequency of a mode, the larger its final amplitude will be. The surprising result is that these modes will probably have a larger amplitude than the 'resonant' mode which has a period equal to the duration of the driver. These properties are illustrated in Figure 2, which plots the normalized r.m.s. amplitude of the natural oscillations of the Alfvén fields following the driven phase as a function of $\omega_{\beta n}$. Contrary to previous studies we do not find the largest response on field lines where $\omega_{\beta n} = 2\pi/\tau_c$, but on field lines for which $\omega_{\beta n} \ll 2\pi/\tau_c$.

The preferential excitement of lower harmonics can be seen experimentally by considering an analogous system: waves on a string. In this system the driving function could be a push at the center of the string. If we push over a short length of the string, then $C_1 \approx C_3 \approx C_5$... (i.e., the driver acts equally on the fundamental, third, fifth, etc. harmonics). The limit of interest is when the periods of the modes are much greater than the period of the push, so it is best to give the string a sharp flick. The analysis suggests that the response will be dominated by the fundamental mode, which will have an amplitude 9 times that of the third harmonic and 25 times that of the fifth harmonic. In practice it certainly seems the case that the dominant mode is the fundamental. (Try it for yourself with a telephone cable!) The explanation for this effect probably lies in the fact that the restoring force due to tension (magnetic tension in Alfvén waves) is greater in higher harmonic modes and inhibits their growth. By flicking the string a quarter of its length from one end a significant second harmonic can be excited; however, the amplitude of the fundamental mode is still $2\sqrt{2}$ times larger than that of the second harmonic.

It is interesting to note that if we model the impulse given to the string not as the smooth pulse given by (19) but by a half cycle of a sine function (i.e., equation (23) with $\ell = 1$), slightly different results are found, although the general behavior of the system is unchanged. In the limit of the duration of the sine impulse ($\pi/\omega_c$) being much less than the eigenperiods of interest ($2\pi/\omega_{\beta n}$), the amplitude of a given mode is $a_{\beta n} \propto C_{n0}/\omega_c\omega_{\beta n}$. Thus flicking a string at its center (so that $C_1 \approx C_3 \approx C_5$...) will mean that the amplitude of the fundamental mode is 3 times that of the third harmonic, and 5 times that of the fifth harmonic: The response of the string is still dominated by the lowest-frequency eigenmodes.

Our predictions for the excitation of Alfvén waves from a pulse can be compared with some recent observations reported by *Lühr et al.* [1990]. They found that the lithium cloud released during the AMPTE mission created a magnetic compression of duration $\tau_c \approx 6$ minutes inside the magnetosphere (estimated from their Figure 4). The small-amplitude oscillations in the ground-based measurements of the $H$ and $D$ magnetic field components before and after the compressional pulse can be used to estimate the natural period of the field lines being observed (see their Figure 3). We estimate this period to be between 3 and 4 minutes, and it probably corresponds to the fundamental period. Thus, from our Figure 2 (or Figure 3a with $\ell = 1$), we anticipate only a small Alfvénic response ($\omega_{\beta n}\tau_c/2\pi = 1.5$ to 2), consistent with the small fluctuations observed subsequent to the compressional pulse.

The final driving function we considered (23) had an intermediate character compared with the previous two driving functions. By specifying the number of half cycles ($\ell$) in the driving function we were able to study the transition from a pulse driver (19), in which the lowest-frequency modes attain the largest amplitudes, to a steady harmonic driver in which the resonant mode attains the largest amplitude. At least 2 complete cycles ($\ell = 4$) of the driver are required before the main Alfvén response is found around the 'resonant' location. However, any eigenmodes with eigenfrequencies similar to the driving frquency will also experience significant excitation. In the toroidal magnetic pulsation problem this would correspond to a broad range of $L$ shells around the resonant $L$ shell being excited. We were able to relate the bandwidth of frequencies in the peak (or the range of $L$ shells excited, if $\omega_{\beta n}(L)$ is a known function) to the number of cycles in the driving function; see equation (27). As one would anticipate, the more cycles the field line is driven over, the higher and thinner the peak around the resonant frequency or resonant $L$ shell.

After having considered the growth of Alfvén fields, we turned in section 4 to the long-term, or asymptotic, solution to these waves. We derived a generalized solution of *Dungey's* [1967] decoupled Alfvén wave modes. It is necessary in this solution that the Alfvén fields be confined to a thin layer. In *Dungey's* solution this was achieved by imposing a large azimuthal wave number ($m$) on the poloidal perturbations. In fact this solution is frequently referred to as the limit of infinite $m$, which is not actually true: The scale of variation across the background field (perpendicular to the Alfvén fields) is $\delta \sim 1/m$. This must be small, but not less than the amplitude of the Alfvén fields, $\epsilon$ [*Wright*, 1990]. The generalized asymptotic solution we present (for $\delta \sim \sqrt{\epsilon}$) includes nonlinear terms and the much smaller non-Alfvénic perturbations which are a necessary part of the solution. The overall picture that emerges is similar to that suggested by *Radoski* [1974], in which all of the fast mode energy is ultimately absorbed by resonant Alfvén waves.

Some of the novel results presented here (e.g., the spatial and temporal variation of the Alfvén fields) may be useful when used to interpret data. For example, the variation of the fields across $L$ shells during the growth of a toroidal pulsation will provide information on how the natural frequencies vary across these $L$ shells. This may yield estimates of the radial plasma density gradient if the background field is known. Information regarding the spatial variation of cavity modes may also be deduced via the magnitude of the integrals $C_n$. For example, if the form of the Alfvén modes is known, we may infer some of the structure of the cavity modes (such as the division of the magnetosphere into 'inner' and 'outer' parts [*Allan et al.*, 1986b; *Zhu and Kivelson*, 1989]). The range of $L$ shells over which the pulsations exist may provide some idea of how many cycles the driving magnetic pressure gradient has executed. The set of coefficients $\{a_{\beta n}\}$ provides a natural Fourier analysis of the temporal variation of the magnetic pressure gradient, and may be used to infer the form of fluctuating or propagating compressional disturbances within the magnetosphere.

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A. Wright, Department of Mathematical and Computational Sciences, University of St. Andrews, St. Andrews KY16 9SS, Scotland.

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