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Magnetic Geometries That Carry Decoupled Transverse or Compressional Magnetic Field Oscillations

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In general, the eigenmodes of a cold inhomogeneous magnetoplasma have magnetic field perturbations with components both parallel and transverse to the background magnetic field direction. Recently, constraints have been derived which the medium must satisfy for the field perturbation to be either entirely compressional, or entirely transverse. Here these criteria are investigated to yield all planar and axisymmetric magnetic field configurations allowing pure decoupled transverse and compressional oscillations. For example, in an axisymmetric dipole field the azimuthal field perturbations decouple exactly, but solely poloidal transverse ones do not. The magnetic field geometry and background plasma density for the most useful decoupled solutions are listed in a table at the end of the paper.

1. INTRODUCTION

One of the simplest ways to investigate the properties of a cold magnetoplasma is to look for coherent, harmonic MHD disturbances of small amplitude. Such an analysis provides information about the characteristic time scales of the system and even its stability. However, in a general medium with spatially varying magnetic fields and plasma density this analysis is formidable because the magnetic field perturbation has components both parallel and perpendicular to the background field. The analysis is simplified considerably by concentrating upon oscillations where the magnetic field perturbation is either quasi-parallel or quasi-perpendicular to the background field [*Fejer*, 1981; *Singer et al.*, 1981; *Chiu*, 1987; *Glassmeier et al.*, 1989]. We shall refer to these modes of oscillation as being compressional and transverse, respectively. The application of decoupled compressional or transverse modes to field fluctuations observed in the terrestrial and jovian magnetospheres has met with considerable success [*Cummings et al.*, 1969; *Singer et al.*, 1981; *Hopcraft and Smith*, 1986; *Glassmeier et al.*, 1989].

When using the decoupled equations, it is obviously important to know how good an approximation this is. For example, the jovian satellite Io launches two (transverse) Alfvén waves into Jupiter's magnetosphere [*Acuña et al.*, 1981; *Barnett*, 1986]. Will these waves remain as a transverse field disturbance, or will they couple to the fast mode and decay? It is well known that in a nonuniform medium a compressional wave will mode convert to a (kinetic) Alfvén wave [*Hasegawa and Chen*, 1976]. Recent work by *Cross* [1988*a*, *b*] derived a restriction that the system must satisfy if it is to support solely transverse field perturbations, namely, that the oscillation frequency must be constant along perturbation magnetic field lines b. Subsequently, *Wright* [1990*a*, *b*] determined constraints on the geometry of the magnetic field, the plasma density distribution, and the form of the medium's boundaries. These latter papers state all the conditions that the medium must satisfy if the transverse and/or compressional oscillations are to decouple from one another exactly. This provides a useful yardstick with which to compare mediums that are of interest to us, for example, dipole or toroidal geometries, stratified or torus-like density distributions, ionospheric or photospheric or even "empty" boundaries.

In this paper we concentrate on the geometrical aspect of the problem. The geometry of a magnetic field may be expressed in terms of three scale factors (see section 2). The geometrical criteria derived in previous studies [*Wright*, 1990*a*, *b*] are given in terms of one or more relations that these factors must satisfy. Given these, it is not always an easy task to write down the corresponding set of field geometries. Indeed, this is the principal aim of our paper. We begin by defining a coordinate system with which to work in section 2. The following three sections derive the most useful geometries permitting pure decoupled transverse and/or compressional oscillations. The results are discussed in section 6, which concludes the main text.

2. COORDINATE SYSTEM

Throughout the paper we make use of a field aligned coordinate system (α, β, γ), where $\hat{\gamma}$ is parallel to the local background magnetic field $\mathbf{B}$. The coordinates α and β label individual field lines since they are constant on any given field line (cf. Euler potentials or Clebsch variables). The background magnetic field is represented in terms of α, β, and γ by

$$\mathbf{B} = f(\alpha, \beta) \cdot \nabla\alpha_{\wedge}\nabla\beta \qquad (1a)$$

$$\mathbf{B} = g(\alpha, \beta, \gamma) \cdot \nabla\gamma \qquad (1b)$$

The transverse coordinates (α, β) may be expressed in terms of matched Euler potentials, in which case f is the Jacobian of this mapping [*Stern*, 1970; *Wright and Smith*, 1990]. An arbitrary magnetic field may always be written at least locally in the form (1*a*), but the functions (α, β) are densely multivalued for stochastic fields and cannot be used to define a global coordinate system (see below). The second relation (1*b*) is always valid in the absence of any background field aligned current. If there is a nonzero background field-aligned current, γ does not form a single-valued global coordinate. We note that similar coordinates have been used

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in previous studies [*Southwood and Hughes*, 1983; *Singer et al.*, 1981; *Murata*, 1986; *Walker*, 1987; *Wright and Smith*, 1990].

To proceed further, we assume that the background field is irrotational and that $\nabla\alpha$ and $\nabla\beta$ are orthogonal (at least throughout the region of space carrying perturbed fields). Without loss of generality the field-aligned coordinate γ is taken as a function of the magnetic scalar potential ψ, so that g is now equal to $\partial\psi/\partial\gamma$ and independent of α and β. The local geometry of the field is contained in the three scale factors h_α, h_β, and h_γ. A real space vector dr is equal to $\hat{\alpha}h_\alpha d\alpha + \hat{\beta}h_\beta d\beta + \hat{\gamma}h_\gamma d\gamma$, and there are well-known expressions for grad, div, and curl in terms of the scale factors [*Davis and Snider*, 1979]. The fact that the background magnetic field $\mathbf{B}$ is solenoidal and irrotational can be stated quite simply (in orthogonal coordinates) as [*Wright*, 1990*a*]

$$Bh_\alpha h_\beta = f(\alpha,\beta) \tag{2a}$$

$$Bh_\gamma = g(\gamma) \tag{2b}$$

For the remainder of this section we try to develop some insight into the field-aligned coordinates by showing how they may be determined.

A suitable point to begin is with the set of two-dimensional planar fields. Let the background magnetic field be independent of the z coordinate and have $B_z = 0$. Each field line is confined to a plane z =const. So, z is a suitable choice for one of the transverse coordinates and we set $\hat{\beta} = \hat{\mathbf{z}}$. The remaining transverse coordinate α is a function of (x, y) describing the shape of the field lines in a surface of constant z. Indeed, this shape coincides with contours of the $\hat{\beta}$ (i.e., $\hat{\mathbf{z}}$) component of the vector potential [cf. *Stern*, 1970]. Thus a general choice for the remaining transverse coordinate is $\alpha = \alpha(A_\beta)$. The field-aligned coordinate is a general function of the magnetic scalar potential, yielding our coordinates $\big(\alpha = \alpha(A_\beta), \beta = z, \gamma = \gamma(\psi)\big)$.

A three-dimensional field geometry that renders relatively straightforward results is the axisymmetric poloidal field. Introducing cylindrical polars (R, ϕ, z), the field is independent of the azimuthal ϕ coordinate, and the azimuthal field component B_ϕ is everywhere zero. Thus the field lines are confined to meridional planes and a suitable choice for one of the transverse coordinates is $\beta = \phi$. The intersection of surfaces of constant α with meridional planes yields the shape of the magnetic field lines. The former surfaces coincide with surfaces of constant $R \cdot A_\phi$ [cf. *Stern*, 1970]. Hence the coordinates may be expressed in terms of the vector and scalar potentials, and the azimuth as $\big(\alpha = \alpha(RA_\phi), \beta = \phi, \gamma = \gamma(\psi)\big)$.

These methods are directly applicable to magnetic fields for which the field lines lie on stationary surfaces. Stochastic magnetic fields, on the other hand, have field lines that pass arbitrarily close to every point in a spatial region. The coordinates (α, β, γ) are then densely multivalued functions on configuration space.

3. TYPE I GEOMETRIES

Wright [1990*a*] uses the linearised induction and momentum equations to investigate the existence of pure decoupled transverse MHD oscillations with perturbed magnetic field component b_α and velocity component u_α. The physical requirements are that b_β, b_γ, and u_β remain zero and that the b_α perturbations along any line (β_0, γ_0) are divergence free. It is shown that these requirements constrain the permissible field geometry thus

$$\frac{h_\alpha}{h_\beta h_\gamma} = P(\alpha,\beta)Q(\beta,\gamma) \tag{3a}$$

where $P(\alpha, \beta)$ and $Q(\beta, \gamma)$ are arbitrary functions. In this type of oscillation the perturbed magnetic field lines are confined to planes of constant β. There is the additional restriction on the background plasma density ρ_0 that

$$\frac{\partial}{\partial\alpha}\left(\frac{B^4}{\rho_0}\right) = 0 \tag{3b}$$

as well as constraints on the boundaries (see *Wright* [1990*a*] for a full discussion and Table 1 for a summary). We denote geometries satisfying (3*a*) as "type I geometries". Decoupling requires the geometry, density, and boundaries to be of the specified forms. Not surprisingly, the (b_α, u_α) wave operators and eigenfrequencies are then independent of α.

Planar Fields

We begin with a planar field $\big(B_x(x,y), B_y(x,y), B_z = 0\big)$ in which the field lines are confined to surfaces of constant z. The first type of transverse mode that is easy to consider is when the perturbation field has only a $\hat{\mathbf{z}}$ component. In order to apply the criterion (3*a*) we define our coordinates with $\hat{\alpha}$ parallel to $\hat{\mathbf{z}}$, i.e., $\big(\alpha = z, \beta = \beta(A_z), \gamma = \gamma(\psi)\big)$. The scale factors are

$$h_\alpha = 1 \qquad h_\beta = \frac{f(\beta)}{B(\beta,\gamma)} \qquad h_\gamma = \frac{g(\gamma)}{B(\beta,\gamma)} \tag{4}$$

which are readily verified to be of form (3*a*). Taking the background plasma density as $\rho_0 = \rho_0(x, y)$ enables (3*b*) to be satisfied and so any planar field can support oscillatory field perturbations confined to the z direction. Such generality is an artifact of two dimensions and does not hold in three dimensions.

The second type of transverse mode that we investigate has field perturbations that are confined to the planes of constant z and so our coordinates are $\big(\alpha = \alpha(A_z), \beta = z, \gamma = \gamma(\psi)\big)$. Exploiting the the homogeneity in z, the general form of the scale factors is

$$h_\alpha = \frac{f(\alpha)}{B(\alpha,\gamma)} \qquad h_\beta = 1 \qquad h_\gamma = \frac{g(\gamma)}{B(\alpha,\gamma)} \tag{5}$$

This yields the simple conclusion that any two-dimensional field $\big(B_x(x,y), B_y(x,y), 0\big)$ carrying oscillatory field perturbations confined to the (x, y) plane is a type I geometry. Defining the plasma density through (3*b*) gives one condition for the existence of a decoupled solution.

Given that the general irrotational and solenoidal background field $\big(B_x(x,y), B_y(x,y), 0\big)$ can support transverse field perturbations both confined and perpendicular to planes of constant z, it is natural to ask if a quite arbitrary transverse perturbation can satisfy (3*a*). This may be investigated by defining a new set of transverse coordinates (α', β') so that the trial field perturbation is always aligned with the local $\hat{\alpha}'$ direction. The details of the calculation are given in Appendix A, where it is shown that (3*a*) cannot be satisfied for arbitrary (α', β'). Some simple geometries that are able to support arbitrarily oriented transverse field perturbations are a uniform background field, a purely toroidal

$\mathbf{B}$, and a field aligned with the cylindrical radial direction $\mathbf{B} = \hat{\mathbf{R}}B_0(a/R)$. (Of course, the field geometry need only be of this form in a subvolume of space that contains perturbed fields and necessarily excludes the z axis.)

Axisymmetric Fields

The simplest field with which to start is axisymmetric and purely toroidal. By identifying the ϕ coordinate with γ and choosing α and β as any orthogonal coordinates in a meridional plane it is evident that the restriction (3a) is fulfilled because of the independence of the metric coefficients of ϕ. Hence this geometry can support transverse oscillations of arbitrary orientation if the density and boundaries are of the permitted forms (see Table 1).

For an axisymmetric and purely poloidal field it is much more difficult to satisfy (3a). The properties of the poloidal field geometry can be found by investigating the cases where the perturbation flux is either completely toroidal (azimuthal) or poloidal. Beginning with the toroidal mode and aligning $\hat{\alpha}$ with $\hat{\phi}$, we take $\left(\alpha = \phi, \beta = \beta(RA_\phi), \gamma = \gamma(\psi)\right)$. Condition (3$a$) is satisfied as all scale factors are independent of azimuth, while (3b) implies $\rho_0 = \rho_0(R, z)$. This confirms the well-known result of, for example, *Dungey* [1967]; azimuthal oscillations decouple exactly in an axisymmetric poloidal background field.

Now consider the axisymmetric poloidal mode. In this case, the field perturbations are confined to meridional planes. To apply the constraint (3a), $\hat{\beta}$ is matched with $\hat{\phi}$ so that $\left(\alpha = \alpha(RA_\phi), \beta = \phi, \gamma = \gamma(\psi)\right)$. The scale factors are given by

$$h_\alpha = \frac{f(\alpha)}{B(\alpha,\gamma)} \cdot \frac{1}{R} \qquad h_\beta = R \qquad h_\gamma = \frac{g(\gamma)}{B(\alpha,\gamma)} \tag{6}$$

which gives

$$\frac{h_\alpha}{h_\beta h_\gamma} = \frac{f(\alpha)}{g(\gamma)} \cdot \frac{1}{R^2} \tag{7}$$

So, a poloidal axisymmetric field can support pure transverse oscillations if and only if R is a separable function of α and γ, i.e.,

$$R^2 = f_\alpha(\alpha) f_\gamma(\gamma) \tag{8}$$

In Appendix B it is shown that the only orthogonal field aligned coordinates satisfying (8) are cylindrical polars, spherical polars, and spheroidals. Surprisingly, these are the only geometries in which the poloidal oscillations can decouple exactly.

Cylindrical and spherical polars are very well known and generate the background magnetic fields $\mathbf{B} = \hat{\mathbf{R}}B_0(a/R)$ and $\mathbf{B} = \hat{\mathbf{r}}B_0(a/r)^2$, respectively. Although the field cannot be aligned with the cylindrical or spherical radial vector throughout all space (as $\mathbf{B}$ must be solenoidal), it certainly can assume this form in a subvolume of space. On the other hand, spheroidal coordinates (λ, ϕ, ν), which are sketched in Figure 1, can be used to define a global magnetic field. Surfaces of constant λ are spheroids of revolution while surfaces of constant ν are two-sheeted hyperboloids of revolution. Magnetic field lines are confined to the latter surfaces, and the three-dimensional structure may be obtained by rotating Figure 1 about its central axis. The coordinates λ and ν are defined as the roots for τ of [*Morse and Feshbach*, 1953]:

$$\frac{R^2}{\tau - a^2} + \frac{z^2}{\tau - b^2} = 1 \tag{9}$$

TABLE 1. The Permitted Decoupled Oscillations in a Variety of Inhomogeneous Media

Type	Perturbations	Background Plasma Density	Boundary Conditions	Field Geometries	Coordinate Alignment
I	Transverse (b_α, u_α)	$b_\alpha(\partial/\partial\alpha)[B^4/\rho_0] = 0$	$b_\alpha(\partial/\partial\alpha)S = 0$	Planar (B_z=0)	z= α or β
				Uniform $\mathbf{B}$	z= z(α,β)
				Cylindrical $\hat{\mathbf{R}}B_0(a/R)$	z= z(α,β)
				Toroidal $\hat{\phi}B_0(a/R)$	z= z(α,β)
				Axisymmetric ($B_\phi = 0$)	$\phi = \alpha$
				Spherical $\hat{\mathbf{r}}B_0(a/r)^2$	$\phi = \alpha$ or β
				Rotational Parabolic	$\phi = \alpha$ or β
				Spheroidal	$\phi = \alpha$ or β
II	Torsional Transverse $(b_\alpha, b_\beta, u_\alpha, u_\beta)$	Arbitrary	$(\hat{\gamma}\wedge\nabla)S = 0$	Uniform $\mathbf{B}$	z= z(α,β)
				Toroidal $\hat{\phi}B_0(a/R)$	z= z(α,β)
				Spherical $\hat{\mathbf{r}}B_0(a/r)^2$	$\phi = \phi(\alpha,\beta)$
	Compressional $(b_\gamma, u_\alpha, u_\beta)$	$b_\gamma(\partial/\partial\gamma)(\rho_0 q^6) = 0$	$\partial S/\partial\gamma = 0$	Uniform $\mathbf{B}$	z= z(α,β)
				Toroidal $\hat{\phi}B_0(a/R)$	z=z(α,β)
III	Compressional (b_γ, u_α)	$(\hat{\alpha}\wedge\nabla)[\rho_0 h_\alpha^2/B^2] = 0$	$(\hat{\alpha}\wedge\nabla)S = 0$; $(\hat{\beta}\wedge\nabla)S_M = 0$	Planar (B_z=0)	z= α or β
				Toroidal $\hat{\phi}B_0(a/R)$	z=z(α,β)
				Axisymmetric ($B_\phi = 0$)	$\phi = \alpha$ or β
				Spheroidal	$\phi = \alpha$ or β

The first column states the type of geometry according to the classification in the text. The nature of the oscillation is given in column 2, along with the nonzero perturbation fields. Certain types of oscillations require a prescribed variation in the plasma density - these are delineated in column 3. The presence of a boundary is described in terms of a surface function $S(\alpha, \beta, \gamma)$ =constant. Such a boundary is termed "massive" or "empty" depending upon whether the plasma density becomes very large or very small on S. (S_M denotes a massive boundary only.) The fifth column lists the appropriate magnetic field geometries for the subclasses of planar magnetic fields ($B_z = 0$) that are invariant in z, and axisymmetric fields ($B_\phi = 0$) that are independent of azimuth (ϕ). The $\hat{\gamma}$ coordinate is always aligned with the local ambient magnetic field direction. The final column details the choice of orientation of the transverse coordinates relative to the invariant coordinate (z or ϕ). For example, consider a transverse oscillation in a Type I geometry: The (b_α, u_α) solution is only valid in a general axisymmetric ($B_\phi = 0$) field if $\hat{\alpha}$ is aligned with $\hat{\phi}$. Whereas the same oscillation in rotational parabolic or spheroidal geometry is permitted when either $\hat{\alpha}$ or $\hat{\beta}$ is aligned with $\hat{\phi}$. Finally, the same oscillation is allowed in a toridal $\hat{\phi}B_0(a/R)$ geometry for an arbitrary orientation of $\hat{\alpha}$ and $\hat{\beta}$.

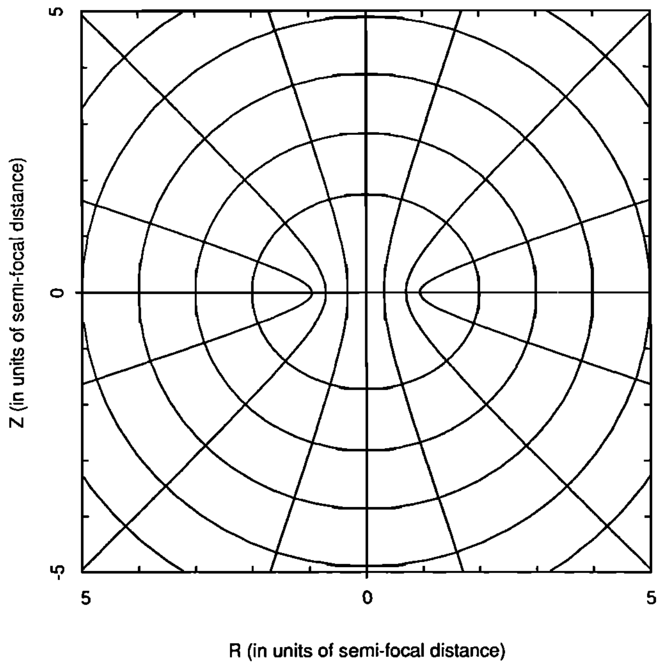

Fig. 1. The figure is sketched in a meridian plane. **B** is aligned with $\hat{\lambda}$. The axisymmetric field can be constructed by rotating the field about the z axis. (In this case the field is supported by a toroidal current ($R > a$) in the plane $z = 0$.)

where a and b are constants and $b^2 \leq \nu \leq a^2 \leq \lambda$. The background field is aligned along $\hat{\nu}$ and the field perturbations along $\hat{\lambda}$ are decoupled. This is a useful configuration for, say, approximating the flux emerging from a sunspot. The complete details of the magnetic field and plasma density for all the solutions constructed in this section are listed in Table 1. Furthermore, we have established an important result; the poloidal oscillations of an axisymmetric poloidal background field do not in general decouple.

4. Type II Geometries

In this section we turn our attention to a different set of geometries, which we shall term "type II geometries". The required form of the scale factors is

$$h_\alpha = p_\alpha(\alpha, \beta) q(\gamma)$$
$$h_\beta = p_\beta(\alpha, \beta) q(\gamma) \qquad (10a)$$

Using the above relations and (2), we find that $h_\gamma = p_\gamma(\alpha, \beta) q_\gamma(\gamma)$, where $p_\gamma = p_\alpha p_\beta / f$ and $q_\gamma = g q^2$. There is an elegant physical interpretation of a magnetic field of this form, namely, the shape of the cross section of any background flux tube is invariant along the length of the tube, since the ratio of h_α to h_β is constant on a given field line.

Type II geometries are suitable for carrying either transverse or compressional magnetic field perturbations. The transverse solution corresponds to a slim tube of background flux supporting a torsional Alfvén wave [*Wright*, 1990*a*]. Thus the perturbation magnetic field lines form closed loops (encircling the background field) whose diameter is much smaller than the scale on which the ambient field varies. There are no restrictions upon the density distribution for this mode of oscillation, but any massive boundary must be perpendicular to the background field to avoid generating a compressional field perturbation. We note that the decoupled transverse equations governing b_α and b_β become identical in geometries of the form defined by (10*a*). Type II geometries can also carry purely compressional field perturbations. This mode is not localized to a slender flux tube but extends across the background field lines [*Wright*, 1990*b*]. The compressional transverse velocity perturbations (u_α, u_β) move the background field lines without bending them and so only generate a parallel field perturbation b_γ. For this mode of oscillation the background plasma density is restricted by

$$\frac{\partial}{\partial \gamma}\left(\rho_0 [q(\gamma)]^6\right) = 0 \qquad (10b)$$

and there are also constraints on any bounding surfaces, as discussed in more detail by *Wright* [1990*b*].

Planar Fields

We begin with the general two-dimensional field $(B_x(x, y), B_y(x, y), B_z = 0)$ and identify β with the z coordinate, which implies that the function q is constant. Returning to the general form of the scale factors for planar fields (4), we see that the field strength must be independent of the γ coordinate to match the form dictated by (10*a*). As a result, h_α does not change along the length of any given field line (i.e., the separation in the $\hat{\alpha}$ direction of a pair of background lines of force will remain constant along their length). Clearly, a uniform background field meets this very restrictive criterion, but are there any other geometries too? The only other possible geometry is a planar axisymmetric toroidal field, where the **B** lines lie on concentric circles.

Axisymmetric Fields

Now let us consider axisymmetric poloidal fields, where $\beta = \phi$ and $B_\phi = 0$ everywhere. The general form of the scale factors for such geometries is stated in (6), and it is evident that the cylindrical radius R must be a separable function of α and γ. We have already derived the set of axisymmetric fields with this property in Appendix B. Of these solutions, only the spherical polar geometry (with associated background field $\mathbf{B} = \hat{\mathbf{r}} B_0 (a/r)^2$) has a scale factor h_α of the form required by (10*a*). A simple physical way to arive at this result is to envisage the shape of the cross section of a background flux tube in the different geometries. Only the radial spherical polar field has the property of preserving the shape along the length of the tube.

5. Type III Geometries

The final mode of oscillation we investigate is purely compressional. Here the only nonzero components of the perturbation magnetic and velocity fields are b_γ and u_α, respectively. We show that this less general compressional mode may be realized in more geometries than the arbitrary $(b_\gamma, u_\alpha, u_\beta)$ mode considered in section 4. Wright [1990*b*] proves that (b_γ, u_α) oscillations are possible only if

$$\frac{h_\gamma}{h_\alpha h_\beta} = P(\alpha) Q(\beta, \gamma) \qquad (11a)$$

which we take as defining "type III geometries". The above requirement is not too restrictive. For example, any field

B that can be represented by true Euler potentials whose gradients are orthogonal (i.e., $f(\alpha,\beta) = \text{const.}$) will satisfy (11$a$). Once again, there are constraints upon the density distribution

$$\left(\frac{\partial}{\partial\beta}, \frac{\partial}{\partial\gamma}\right)\left[\frac{B^2}{h_\alpha^2\rho_0}\right] = 0 \tag{11b}$$

and the massive bounding surfaces compatible with this mode (see Table 1).

Planar Fields

As in section 3, our coordinates are $\left(\alpha = \alpha(A_z), \beta = z, \gamma = \gamma(\psi)\right)$. It is straightforward to use (4) to deduce

$$\frac{h_\gamma}{h_\alpha h_\beta} = \frac{g(\gamma)}{f(\alpha)} \tag{12}$$

Any two-dimensional field $\left(B_x(x,y), B_y(x,y), 0\right)$ can support compressional oscillations with a perturbation velocity field in the (x,y) plane. Equally, by taking coordinates such that $\left(\alpha = z, \beta = \beta(A_z), \gamma = \gamma(\psi)\right)$ we find that compressional oscillations with perturbation velocity parallel to $\hat{\mathbf{z}}$ are also permissible. This augments the results of section 3 – any planar field is a type III geometry and permits decoupled oscillations if the density is of form (11b).

Axisymmetric Fields

Toroidal axisymmetric fields allow compressional oscillations of arbitrary orientation. This follows immediately from the independence of the scale factors of azimuth. More surprising, in view of the restrictive results of section 3, is that both toroidal and poloidal compressional oscillations can decouple exactly in an axisymmetric poloidal background field.

Turning to poloidal **B** fields, let us examine the toroidal (b_γ, u_ϕ) mode by taking $\left(\alpha = \phi, \beta = \beta(RA_\phi), \gamma = \gamma(\psi)\right)$. Condition (11$a$) is satisfied as all scale factors are independent of azimuth. For the poloidal (b_γ, u_α) mode, $\hat{\beta}$ is matched with $\hat{\phi}$ so that $\left(\alpha = \alpha(RA_\phi), \beta = \phi, \gamma = \gamma(\psi)\right)$. The scale factors are given in (6), and we readily find

$$\frac{h_\gamma}{h_\alpha h_\beta} = \frac{g(\gamma)}{f(\alpha)} \tag{13}$$

which is of the required form. This generality is particularly astonishing in view of the scarcity of axisymmetric geometries permitting decoupling of transverse oscillations. It should be noted, however, that the exclusion of the perturbations from the currents that generate the ambient field will necessitate the introduction of boundaries. Since these boundaries may not lie across background field lines (see Table 1) it may be difficult to find suitable boundary surfaces for some examples.

In Table 1 the details of the magnetic fields and background plasma densities permitting decoupled modes of oscillation are collected. This summarizes the most important results derived in the last three sections.

6. Discussion and Conclusions

When confronted with a nonuniform magnetoplasma, a preliminary investigation of the system often involves considering the quasi-transverse and quasi-parallel field perturbations independently [e.g., *Glassmeier et al.*, 1989; *Singer et al.*, 1981]. Whilst this is an instructive step, it is important to know to how good approximation any given mode is truely decoupled from other modes. For example, in an axisymmetric dipolar background magnetic field the azimuthal field perturbations decouple exactly [*Dungey*, 1967]. However, in the same geometry the purely poloidal field perturbations do not [*Cross*, 1988a; *Wright*, 1990a]. Evidently, one may anticipate that the azimuthal field perturbations will be longer lived than their poloidal counterparts (which will probably couple to a compressional mode and decay more rapidly).

Transverse azimuthal perturbations are frequently observed in planetary magnetospheres. The excitation mechanism will affect the symmetry of such oscillations. For example, ion pick-up from the Io torus in the Jovian magnetosphere provides, to lowest order, axisymmetric excitation [*Glassmeier et al.*, 1989] and can generate perturbations with a vanishing azimuthal wave number naturally. On the other hand, in the terrestrial magnetosphere transverse azimuthal field perturbations can be excited by nonaxisymmetric compressions at the magnetopause [*Inhester*, 1987]. Although such a resonant coupling cannot be described by our equations, the spatial variation of the eigenmodes along the field lines is identical in both calculations to lowest order.

What happens when one calculates, for example, the decoupled transverse modes of a medium which is not of the required form? *Fejer* [1981] notes that a paradox may arise if a disturbance is expressed in terms of the two decoupled transverse modes (termed "toroidal" and "poloidal" in his paper): The state of the perturbed field may be constructed at later times by considering only one of the set of transverse modes - the other transverse field perturbation may be inferred from $\nabla \cdot \mathbf{b}_\perp = 0$. The paradox arises because, in general, the inferred solution at later times depends upon which set of modes is used at the outset.

The use of the word "decoupled" is something of a misnomer in this situation. In fact, the arbitrary (and often unphysical) setting of the compressional field perturbation to zero actually couples the two transverse field perturbations via $\nabla \cdot \mathbf{b}_\perp = 0$ [*Fejer*, 1981; *Wright*, 1990a]. Mediums incorporating a type II magnetic field geometry have the special property that b_α and b_β may evolve "independently" of one another without violating the solenoidal restriction at later times. Type II geometries guarantee that a torsional Alfvén wave will not generate a parallel field perturbation $b_\parallel$ (if the boundaries are also suitable). It is interesting to consider the fate of a torsional Alfvén wave when the medium is not suitable for its survival: *Fejer* [1981] suggests that mismatched transverse modes will lead to a breakdown of the assumption that $b_\parallel = 0$. Alternatively, *Chiu* [1987] insists upon keeping $b_\parallel = 0$ and invokes the generation of "parasitic" transverse modes that maintain $\nabla \cdot \mathbf{b}_\perp = 0$ at later times. If the medium only departs slightly from the required ideal form, then there may develop a relatively small $b_\parallel$. One possible result would be a coupling to the fast mode which would tend to propagate isotropically [*Walker*, 1987]. In these circumstances it is possible that the disturbance would appear to be a torsional Alfvén wave that slowly decays in amplitude as a result of coupling with the (compressional) fast mode. On the other hand, *Cross* [1988a, b] has found Alfvén wave solutions with a finite $b_\parallel$. In this circumstance there is an energy flow across **B**, but this amounts to no more than a circulation of energy around background lines of force.

In addition to quasi-transverse modes, magnetospheric physicists are also interested in compressional modes, for example, the transmission of a compressional impulse from the outer to the inner magnetosphere. If the background field is taken to be that of an axisymmetric dipole, we know that this (type III) geometry is suitable for carrying a purely compressional disturbance. However, the boundaries and density distributions found in planetary magnetospheres are not of the required form [*Wright*, 1990*b*]. As a result, we expect a coupled compressional/transverse oscillation. Indeed, such a mode has been observed in the Jovian magnetosphere [*Glassmeier et al.*, 1989] where the compressional and transverse poloidal field perturbation oscillate coherently. This could be anticipated on physical grounds from the solenoidal nature of **b** and the fact that $b_{\|} \neq 0$ at the equator but must vanish at the ionospheric footpoints.

The aim of this paper has been to present the most useful magnetic geometries for which some type of completely decoupled mode may exist. Often we may be interested in the properties of a field that has a geometry which is unsuitable for decoupled modes. In this situation, some idea of how realistic quasi-transverse or quasi-parallel modes are may be gained by seeing how much the background medium departs from one of the 'ideal' mediums in this text. This is perhaps the most interesting direction in which to continue this work.

One approach is to investigate the strength of coupling between transverse and compressional field perturbations. For example, the models given in this paper can be used as a lowest-order solution which can be perturbed by appropriately small changes in the plasma density (ρ) or the field geometry (h,s). Alternatively, a multiple-scale analysis of the perturbed fields may prove useful to investigate systems where there are weak gradients of the perturbations (for example, low azimuthal wave number transverse toroidal field perturbations in a background poloidal field). In contrast to weak gradients, the existence of strong gradients occurs naturally in resonance problems [*Allan et al.*, 1985; *Inhester*, 1986; *Inhester*, 1987], and boundary layer techniques often prove useful. Besides these analytical tools, numerical methods can provide solutions that are otherwise untractable.

Our work can be extended in a different direction by relaxing the assumption of ideal Ohm's law. Including nonideal effects such as finite resistivity or kinetic effects will permit finite $\mathbf{E}_{\|}$, for example, as in the kinetic Alfvén wave. Despite this wave being transverse the group velocity will have a small perpendicular component. Such calculations can be particularly relevant to laboratory plasmas.

APPENDIX A

In this appendix we investigate the ability of a general planar field $\left(B_x(x,y), B_y(x,y), B_z = 0\right)$ to carry arbitrary transverse oscillations. A suitable way to pose the problem is to introduce new transverse coordinates, leaving the field-aligned coordinate remaining unchanged; $\left(\alpha' = \alpha'(\alpha,\beta), \beta' = \beta'(\alpha,\beta), \gamma' = \gamma\right)$. The new coordinate β' is chosen so that the perturbation field line lies in the surface β' =const. The vector $\hat{\beta}'$ is equal to $h_{\beta'}\nabla\beta'$ and so

$$\hat{\beta}' = h_{\beta'}\cdot\left[\frac{\partial\beta'}{\partial\alpha}\cdot\nabla\alpha + \frac{\partial\beta'}{\partial\beta}\cdot\nabla\beta\right] \equiv h_{\beta'}\cdot\left[\frac{\hat{\alpha}}{h_\alpha}\cdot\frac{\partial\beta'}{\partial\alpha} + \frac{\hat{\beta}}{h_\beta}\cdot\frac{\partial\beta'}{\partial\beta}\right] \tag{A1}$$

In order to determine the function $\alpha'(\alpha,\beta)$, which provides an orthogonal coordinate to $\hat{\beta}'$, we take the vector product $\hat{\alpha}' = \hat{\beta}' \wedge \hat{\gamma}'$ to get

$$\hat{\alpha}' = \hat{\alpha}\cdot\frac{h_{\beta'}}{h_\beta}\frac{\partial\beta'}{\partial\beta} - \hat{\beta}\cdot\frac{h_{\beta'}}{h_\alpha}\frac{\partial\beta'}{\partial\alpha} \tag{A2}$$

Alternatively, $\hat{\alpha}'$ may be expanded in a form similar to (A1)

$$\hat{\alpha}' = h_{\alpha'}\cdot\left[\frac{\hat{\alpha}}{h_\alpha}\cdot\frac{\partial\alpha'}{\partial\alpha} + \frac{\hat{\beta}}{h_\beta}\cdot\frac{\partial\alpha'}{\partial\beta}\right] \tag{A3}$$

Whilst we can be assured that the form for $\hat{\alpha}'$ given in (A2) will complete a right-handed triad, there is no such restriction upon the expression (A3). Orthogonality can be effectively imposed upon the latter relation by equating its $\hat{\alpha}$ and $\hat{\beta}$ components with those of (A2). Matching the $\hat{\alpha}$ components (assuming $\partial\alpha'/\partial\alpha$ is nonzero) gives

$$\frac{h_{\alpha'}}{h_{\beta'}h_{\gamma'}} = \left[\frac{h_\alpha}{h_\beta h_\gamma}\right]\cdot\frac{\partial\beta'/\partial\beta}{\partial\alpha'/\partial\alpha} \tag{A4}$$

We require that the left hand side (lhs) is of the general form $P'(\alpha',\beta')Q'(\beta',\gamma')$, (cf. equation (3)). Given the old coordinate system (in which β is taken as z), we know that the term in square brackets may be written as $P(\alpha)Q(\gamma)$ because of invariance in the β (*not* β') coordinate. Noting that the last term in (A4) is a function of α' and β', and that α depends only upon α' and β', it follows that (A4) can always be be written in the required form. Equating $\hat{\beta}$ components of (A2) and (A3) gives (assuming $\partial\alpha'/\partial\beta$ is nonzero)

$$\frac{h_{\alpha'}}{h_{\beta'}h_{\gamma'}} = -\left[\frac{h_\alpha}{h_\beta h_\gamma}\right]\cdot\frac{h_\beta^2}{h_\alpha^2}\cdot\frac{\partial\beta'/\partial\alpha}{\partial\alpha'/\partial\beta} \tag{A5}$$

Once again, we require (A5) to be of the general form $P'(\alpha',\beta')Q'(\beta',\gamma')$. As before, the square-bracketed term and the final term are consistent with this expression. However, the additional term in the middle of the right hand side (rhs) is more problematic. It may be written as $\left[B(\alpha,\gamma)/f(\alpha)\right]^2$, where we have used (2$a$), invariance in β, and $h_\beta = 1$. Since we have $\alpha(\alpha',\beta')$ and $\gamma(\gamma')$, (A5) is only of the required form if B is independent of either α' or γ'. Such a requirement rules out most field geometries. Some planar geometries for which the field strength is independendent of γ' are a uniform field, or a purely toroidal field. If a subvolume of space has the background magnetic field aligned with the cylindrical radial vector this geometry will satisfy (3) in an arbitrary primed system since the field strength will always be in dependent of α'.

Note that if our assumptions of $\partial\alpha'/\partial\alpha$ and $\partial\alpha'/\partial\beta$ being nonzero are not true, then $\hat{\alpha}'$ will coincide with either $\hat{\alpha}$ or $\hat{\beta}$ from the original coordinate system and we shall be considering one of the modes already discussed in section 3.

APPENDIX B

In this appendix we derive the set of axisymmetric geometries for which the cylindrical radius R is a separable function of α and γ. The metric of the axisymmetric field aligned coordinate system is

$$ds^2 = h_\alpha d\alpha^2 + h_\beta^2 d\beta^2 + h_\gamma^2 d\gamma^2 \tag{B1}$$

where β is identified with the ϕ coordinate and $h_\beta^2 = R^2 = x^2 + y^2$. We wish to find all geometries satisfying

$$R^2 = f_\alpha(\alpha) f_\gamma(\gamma) \tag{B2}$$

where f_α and f_γ are arbitrary functions of indicated argument. First, we must apply the condition that the space is flat, i.e., the Riemann tensor is zero [*Synge and Schild*, 1956]. The nonvanishing components of the Riemann tensor yield the coupled partial differential equations

$$\frac{\partial \log h_\beta^2}{\partial \alpha}\frac{\partial \log h_\beta^2}{\partial \gamma} - \frac{\partial \log h_\beta^2}{\partial \alpha}\frac{\partial \log h_\alpha^2}{\partial \gamma} - \frac{\partial \log h_\beta^2}{\partial \gamma}\frac{\partial \log h_\gamma^2}{\partial \alpha} = 0 \tag{B3}$$

$$\frac{1}{h_\alpha^2}\left(\frac{2\partial^2 \log h_\beta^2}{\partial \alpha^2} + \frac{\partial \log h_\beta^2}{\partial \alpha}\frac{\partial}{\partial \alpha}\log\frac{h_\beta^2}{h_\alpha^2}\right) + \frac{1}{h_\beta^2}\left(\frac{2\partial^2 \log h_\alpha^2}{\partial \beta^2}\right. \\ \left. + \frac{\partial \log h_\alpha^2}{\partial \beta}\frac{\partial}{\partial \beta}\log\frac{h_\alpha^2}{h_\beta^2}\right) + \frac{1}{h_\gamma^2}\frac{\partial \log h_\beta^2}{\partial \gamma}\frac{\partial \log h_\alpha^2}{\partial \gamma} = 0 \tag{B4}$$

where all cyclic combinations are taken. If the vector fields $\nabla\alpha$, $\nabla\beta$, and $\nabla\gamma$ are to mesh to give smooth and mutually orthogonal surfaces, then we must additionally impose the integrability condition [*Schutz*, 1980]:

$$\frac{\partial^2 \log h_\alpha^2}{\partial \alpha \partial \gamma} + \frac{\partial \log h_\alpha^2}{\partial \gamma}\frac{\partial \log h_\gamma^2}{\partial \alpha} = 0 \tag{B5}$$

Using (B3) and (B5), it is straightforward (but lengthy) to show that the scale factors are of the form

$$h_\alpha^2 = f_\alpha + f_\gamma \tag{B6a}$$

$$h_\beta^2 = f_\alpha f_\gamma \tag{B6b}$$

$$h_\gamma^2 = f_\gamma + f_\alpha \tag{B6c}$$

where f_α is at most a function of α and likewise for f_γ. The investigation may usefully be divided into three cases.

If f_γ is a constant, then the coordinates may be rescaled to give

$$h_\alpha^2 = \psi_1^2(\alpha) \qquad h_\beta^2 = \psi_2^2(\alpha) \qquad h_\gamma^2 = 1 \tag{B7}$$

Imposing the so far unused three equations (B4) gives the solution without loss of generality as $\psi_1 = 1$ and $\psi_2 = \alpha$, and so the coordinates are recognised as cylindrical polars. An exactly analogous method may be used when f_α is constant to generate spherical polars.

Finally, if neither f_α nor f_γ are constant, substituting (B6) into (B4) gives the PDEs

$$\begin{aligned} 2f_\alpha f_\alpha'' - {f_\alpha'}^2 &= c f_\alpha^2 \\ \left(\frac{{f_\gamma'}^2}{f_\gamma}\right)' &= -c f_\gamma' \end{aligned} \tag{B8}$$

which can be integrated to yield

$$\begin{aligned} {f_\alpha'}^2 &= c f_\alpha^2 + d f_\alpha \\ {f_\gamma'}^2 &= -c f_\gamma^2 + d f_\gamma \end{aligned} \tag{B9}$$

When $c = 0$, the coordinates are the rotational parabolic system [*Morse and Feshbach*, 1953]. In the general $c \neq 0$ case, straightforward rescaling gives the spheroidal coordinates defined in (9). Hence we have proven that the only axisymmetric coordinates for which R is of the given form (B2) are cylindrical polar, spherical polar, rotational parabolic, and spheroidal. These are the axisymmetric confocal quadric surfaces first written down by *Eisenhart* [1934]. Although rotational parabolics formally satisfy (B2), they cannot be used to define a sensible global magnetic field. The applications of the other coordinate systems are discussed in section 3.

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