STATIONARY ALFVÉNIC STRUCTURES

A. N. Wright¹ and D. J. Southwood

Physics Department, Imperial College, London

Abstract. A nonlinear, analytical model is developed to describe equilibrium Alfvén wave structures such as that associated with flow about a conducting obstacle in a flowing plasma. Firstly, in order to give physical insight into the processes involved, we discuss two worked examples one of which extends previous models of the Alfvén wave wake produced by the Jovian satellite Io. We describe in detail the fields and, in particular, the current systems associated with the model. Finally, we present a general formalism which leads to a general solution that should be applicable in modeling other incompressible space plasma flows.

1. Introduction

This paper concerns the manner in which current may be carried away from an object in a streaming plasma. The only hydromagnetic wave mode that carries current away from source is the intermediate, or Alfvén, mode [Dungey, 1967] and our attention here concentrates on that mode. In particular, we are interested in its current-carrying properties. Neubauer [1980] has produced a nonlinear solution for a stationary Alfvén wave standing in a background flow. His work was specifically applied to the problem of plasma flow about the Jovian moon Io, but nevertheless the work forms the starting point for our study.

Consider an isolated source of current in a plasma. Charge may not accumulate and thus the plasma must carry a current to remove it either to a sink located within the plasma or to the boundary. As long as the magnetic field lines are perfectly conducting, the evident route for the current is along the field. In a stationary plasma, a transient source of charge would give rise to an associated Alfvén wave that would carry the corresponding field-aligned current. In a streaming plasma, a current source would also exicte Alfvén waves propagating away from the source in each direction along the field. These would however propagate in the plasma rest frame and, thus, in the source frame, move along the characteristic directions $V^{\pm} = V_C \pm V_A$, where V_C is the background flow velocity and V_A is the background Alfvén velocity.

In fact the net current carried by the Alfvén waves in the source frame is along these characteristic directions rather than the field direction itself [Neubauer, 1980]. Indeed as the directions are unperturbed by the wave even in the nonlinear case the net current flows along directions determinable from the unperturbed flow and field. However,

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Paper number 6A8516 0148-0227/87/006A-8516\$05.00 although the net current flow is along the characteristic directions, the wave can create currents transverse to the characteristic directions that serve to redirect the flow in the vicinity of the wave. In this paper we examine current flow patterns set up within the wave and relate them to the nature of the characteristic current flow and the electric and magnetic field patterns of the wave. The perpendicular current system and its closure have received very little attention to date. Its existence has been pointed out by Southwood and Dunlop [1984] and Rasmussen et al. [1985]. However, the basic topology of these currents was not understood in previous work.

We use the term "stationary Alfvénic structures" to describe current-carrying Alfvén waves standing in a flow. Two simple cases are drived in detail early in the paper in an effort to give some insight into the processes involved in maintaining an equilibrium structure. The first example is one in which there is a net current flow from the source emitting the waves. One potential application would be to the Birkeland current systems associated with geomagnetic substorms in the terrestrail magnetosphere. Another might be the twisted field structures detected in flux transfer events at the terrestrial magnetopause [Saunders et al., 1984]. The flux ropes in the Venus ionosphere [Elphic and Russell, 1983] carry a net current, and Alfvén waves may play a part in their generation. The ropes themselves are detected deep within the ionosphere and, being highly nonuniform, are not as simple as the structures to which we restrict ourselves in this paper.

Our second example has a bipolar current system at its core and is similar to several models of the production of Alfvén waves by the Jovian satellite Io [Neubauer, 1980; Southwood et al., 1980; Goertz and Deift, 1973]. An early treatment of the basic mathematical problem is that of Drell et al. [1965] who considered the production of Alfvén waves by currents flowing in an artificial satellite moving through the ionosphere and the drag associated with such an interaction. Recently, Rasmussen et al. [1985] have described various shaped conductors moving through a magnetoplasma.

Working from the particular to the general, we build on the examples presented to develop a general solution expressed as a superposition of simple sources. A general formalism is developed in which the wave fields are derived from two stream functions. The stream functions are simply related (one is often the derivative of the other). A consequence of this property is that the source structure of the perpendicular current has a polar order twice that of the source pattern of magnetic field, plasma flow or aligned current fields.

2. Stationary Alfvénic Structures

In this section we discuss the geometry and orientation of the perturbations to electric (E) and magnetic (b) fields, the plasma flow velocity (u) and the current density (j) associated with stationary

¹Now at Theoretical Astronomy Unit, School of Mathematical Sciences, Queen Mary College, London E1 4NS, England.

Alfvénic structures. The different propagation speeds of the three MHD wave modes suggest that as the waves propagate they will become spatially separate unless there is some local production. This results in the magnetosonic modes separating from the Alfvén mode and, since it is the magnetosonic modes that produce changes in pressure, we anticipate constant plasma pressure and density throughout the Alfvénic stuctures that we consider in this paper.

Let us consider the equations governing the flow. Both the magnetic field, **B**, and plasma flow velocity, V, are solenoidal within an isodensity (i.e., constant plasma pressure) flow,

$$\nabla \mathbf{B} = \nabla \mathbf{V} = \mathbf{0} \tag{1}$$

The magnetohydrodynamic equations that the magnetic and flow velocity fields must satisfy are the momentum equation,

$$\rho(\mathbf{V}.\nabla)\mathbf{V} = (\mathbf{B}.\nabla)\mathbf{B}/\mu_0 - \nabla(\mathbf{p}+\mathbf{B}^2/2\mu_0)$$
(2)

where ρ is the plasma mass density; the frozen field condition is

$$(\mathbf{V}.\nabla)\mathbf{B} - (\mathbf{B}.\nabla)\mathbf{V} \tag{3}$$

The incompressibility condition, (1) implies that the mass density is constant. If the total pressure is constant also, it follows by adding and subtracting equations (2) and (3) that there is a general solution for flow and field of the form

$$V \pm \frac{B}{(\mu_0 \rho)^{\frac{1}{2}}} - \text{const} - V_c \pm \frac{B_0}{(\mu_0 \rho)^{\frac{1}{2}}}$$
 (4)

where V_A is the Alfvén speed, in which the field and plasma pressure variation is controlled by the requirement that the total pressure be constant,

$$p + B^2/2\mu_0 = const$$
 (5)

(see, for instance, Dungey [1967] and Cowling [1976]). We shall actually maintain the plasma and field pressure constant independent of one another since the plasma density is constant.

The solution can be checked by simple substitution. A similar solution was given by Neubauer [1980]. Note that, if the wave is embedded in a uniform flow, V_c , and magnetic field, B_0 , the perturbation flow, u, and field, b, satisfy the Walén, or Alfvén, relation [Alfvén, 1942; Walén, 1944]

$$\mathbf{b} = \pm \mathbf{u}(\mu_0 \rho)^{\frac{1}{2}} \tag{6}$$

Electric Fields

Let us now consider the electric field configuration implicit in a stationary Alfvénic structure. Following Neubauer [1980], one can compute

$$\nabla . (\mathbf{E} + \mathbf{E}_{\mathbf{C}}) = -\nabla . (\mathbf{V}_{\Lambda} \mathbf{B}) = \mathbf{V} . (\nabla_{\Lambda} \mathbf{B}) = \mathbf{B} . (\nabla_{\Lambda} \mathbf{V})$$
(7)

where E is the wave, or perturbation, electric field and E_c is the convection electric field. Substituting from the Alfvén relation, (6), one finds that

$$\nabla \cdot \mathbf{E} = (\mathbf{V}_{\mathbf{C}} \pm \mathbf{V}_{\mathbf{A}}) \cdot (\nabla_{\mathbf{A}} \mathbf{B})$$
(8)

and thus

$$\nabla \mathbf{E} = \mu_0 (\mathbf{V}_{\mathbf{C}} \pm \mathbf{V}_{\mathbf{A}}) \cdot \mathbf{j}$$
⁽⁹⁾

As the fields are stationary the wave electric field can be represented by a potential

$$\mathbf{E} = -\nabla \Phi \tag{10}$$

and thus

$$\nabla^2 \Phi = -\mu_0 (\mathbf{V}_{\mathbf{c}} \pm \mathbf{V}_{\mathbf{A}}) \cdot \mathbf{j} = -\mu_0 \mathbf{V}^{\pm} \cdot \mathbf{j}$$
(11)

Evidently the upper and lower signs in the foregoing correspond to waves propagating parallel and antiparallel to the ambient field in the plasma frame of reference. In the stationary frame the vectors, V^{\pm} represent the wave characteristic directions. The directions are fixed in space at all points where the wave fields are present by virtue of equation (4). There is no loss of generality by specializing at this stage and taking only the lower sign (wave propagating antiparallel to B_0 in the plasma rest frame). As the wave propagates along the characteristic direction, in the presence of a steady source, the solution is independent of the coordinate parallel to a direction fixed by the constant vector, V^{\pm} . Thus all wave fields vary only in planes perpendicular to the characteristic.

In particular, the invariance along the characteristic means that the potential satisfying (11) is a function only of the coordinates perpendicular to the characteristic and from (10) it follows that the electric field is at right angles to the characteristic. The electric field must also be perpendicular to the total magnetic (background plus wave) field. The relationship with the characteristic may be seen explicitly by writing the wave and background fields separately. Using the MHD Ohm's law, one has

$$\mathbf{E} + \mathbf{E}_{\mathbf{C}} = -\mathbf{V}_{\mathbf{A}}\mathbf{B} \tag{12}$$

where the background convection electric field is $E_c = -V_c \wedge B_0$. Thus the perturbation (wave) electric field is

$$\mathbf{E} = -\mathbf{u}_{\Lambda}\mathbf{B}_{0} - \mathbf{V}_{C\Lambda}\mathbf{b} - \mathbf{u}_{\Lambda}\mathbf{b}$$
(13)

By using (6), it is a matter of simple manipulation for one to show that

$$\mathbf{\underline{E}} + \mathbf{E}_{\mathbf{c}} = -\mathbf{V}_{\mathbf{A}}^{-}(\mathbf{B}_{\mathbf{0}} + \mathbf{b}) = -\mathbf{V}_{\mathbf{A}}^{-}\mathbf{B}$$
(14)

Thus the field is perpendicular to both total magnetic field and the characteristic.

The Magnetic Polarization and the Current Flow

From now on, we shall take the direction of the characteristic to be the z axis (see Figure 1). Fields and currents will thus be independent of z. Equation (11) shows the central importance of the current density along the characteristic. Without current flow somewhere parallel to the characteristic direction there will be no wave fields. In reality at some point the characteristic-aligned current must attach to the source where the wave is set up. Although the current parallel to the characteristic is solenoidal, it is not the only current system in the wave. Consider the following argument. The Ampere circuital relation shows one that a current in the z direction gives rise to magnetic field components in the (x,y) plane perpendicular to the characteristic. However, a unidirectional current flow cannot give rise to a



Fig. 1. The production of a stationary Alfvénic structure: The ambient plasma flow (V_c) is supported by the convection electric field, $E_c = -V_{CA}B_0$. The Alfvénic perturbations are guided along the magnetic field and drift past the source forming a "wing." The Alfvén Mach number is equal to $\tan \theta_A = V_c/V_A = M_A$.

magnetic component parallel to itself. Any such component requires current flow perpendicular to the characteristic. From Ampere's law, the perpendicular currents are given by

$$\mu_{0}\mathbf{j}_{\mathbf{x}} - \frac{\partial \mathbf{b}_{\mathbf{z}}}{\partial \mathbf{y}} \qquad \mu_{0}\mathbf{j}_{\mathbf{y}} - - \frac{\partial \mathbf{b}_{\mathbf{z}}}{\partial \mathbf{x}} \tag{15}$$

and the imposition of constant magnetic field pressure implies that

$$b_{z} = [B_{0}^{2} - (b_{y} - B_{0} \sin \theta_{A})^{2} - b_{x}^{2}]^{\frac{1}{2}} - B_{0} \cos \theta_{A}$$
(16a)

$$b_z = b_v tan \theta_A$$
 (16b)

where θ_A is the angle between V^- background field, B_0 . Equation (16) and the Equation (16b) is the small-amplitude limit of (16a) and, as we shall see, is useful when considering the form of the perpendicular currents. Equation (16a) is the exact nonlinear solution. Equation (16b) is equivalent to requiring that the magnetic field perturbation be orthogonal to the background magnetic field. Since the disturbances we are interested in decrease in amplitude as we move away from the stationary will the nonlinear solution be structure, well-approximated by the linear expressions once we are a few scale lengths away from the center of the strucure. (The maximum size of the perturbation is twice the background field strength.)

We have now in principle all the information required to determine the distribution of the current in the structure. Recalling that the characteristic-aligned current is independent of z and thus reflects the current-emitting properties of the source which may be of limited extent, we shall find it useful to distinguish the regions of (x,y) where there is current flow parallel to the characteristics. Often in simple models this current is taken to flow in sheets, the so-called Alfven wings [Drell et al., 1965]. Outside the wings the current flow is two dimensional and in the (x,y) plane.

There are several equivalent ways of picturing the self-consistent field structures. For example equation (11) shows that distribution of j also reflects the charge distribution in (x,y) which in turn can be thought of as determining the electrostatic potential and the electric field. The plasma flow perturbation (u) follows from equation (12), and the magnetic field perturbation follows from equation (4). Alternatively, using the current in itself leads one directly to the form of the magnetic field via Ampere's law.

Note how the method of characteristics used by Neubauer reveals the importance of characteristic-aligned current rather than field-aligned current in steady wave structures. In other treatments the presence of field-aligned current flow has been emphasized [e.g., Southwood et al., 1980]. Field-aligned current flow characterizes the intermediate MHD mode and furthermore is a necessary feature of perpendicular stress transfer along the field direction [e.g., Southwood and Hughes, 1983]. Equation (4) shows that there must be a component of the characteristic current flow along the ambient field direction.

In the following two sections we give examples of stationary Alfvénic structures and attempt to give physical insight into the mechanics of each system.

3. Monopolar J_z

A undirectional characteristic-aligned current with cylindrical cross section in the (x,y) plane is one of the simplest forms of distribution of characteristic-aligned current to envisage. The source is a net current emitter assumed to be located far away in the positive z direction. There are several



Fig. 2. Current flow in the (x,y) plane for a monopolar j_z .

possible instances of such sources in the solar system. At Venus, twisted magnetic field structures are detected in the interior of the ionosphere. These are known as flux ropes. Almost certainly there is a net current flux along the structure although their formation and origin is not understood [Elphic and Russell, 1983]. At the earth's magnetopause, flux transfer events are a common phenomenon [Rijnbeek et al., 1984; Berchem and Russell, 1984]. There is a central core current in such structures [Cowley, 1982; Paschmann et al., 1982; Saunders et al., 1984]. Similar wound- up field structures are believed to occur on the sun [e.g. Priest, 1982].

Let us now use cylindrical polar coordinates polar (r,φ,z) and consider the following form for j_z ;

.

$$\mathbf{j}_{\mathbf{Z}} = \begin{cases} \mathbf{j}_{\mathbf{Z}_0} & \mathbf{r} < \mathbf{r}_0 \\ \mathbf{0} & \mathbf{r} > \mathbf{r}_0 \end{cases}$$
(17)

i.e., the current density is uniform inside the cylinder $r = r_0$, $(r^2 = x^2 + y^2)$.

The electric field and potential of the disturbance are given from equation (11) and (10).

$$E_{r} = \begin{cases} \frac{1}{2}\mu_{0} \ V^{-} \ j_{Z0} \ r & r < r_{0} \\ \frac{1}{2}\mu_{0} \ V^{-} \ j_{Z0} \ (r_{0}^{2}/r) & r > r_{0} \end{cases}$$
(18)

$$\Phi = \begin{cases} -\frac{1}{4}\mu_0 \ V^- \ \mathbf{j}_{\mathbf{Z}0} \ (\mathbf{r}^2 - \mathbf{r}_0^2) & \mathbf{r} < \mathbf{r}_0 \\ -\frac{1}{4}\mu_0 \ V^- \ \mathbf{j}_{\mathbf{Z}0} \ \mathbf{r}_0^2 \ \ln(\mathbf{r}/\mathbf{r}_0) & \mathbf{r} > \mathbf{r}_0 \end{cases}$$
(19)

The current and the azimuthal magnetic field are related by the z component of Ampere's law. One has

$$\mathbf{b}_{\varphi} = \begin{cases} \frac{1}{2}\mu_0 \ \mathbf{j}_{\mathbf{Z}0} \ \mathbf{r} & \mathbf{r} < \mathbf{r}_0 \\ \frac{1}{2}\mu_0 \ \mathbf{j}_{\mathbf{Z}0} \ \mathbf{r}_0^2/\mathbf{r} & \mathbf{r} > \mathbf{r}_0 \end{cases}$$
(20)



Fig. 3. The closure of total current: one of the family of dipole curves is shown to form a cascade along the length of the Alfvén wing. There is a similar cascade on the backside of the cylinder.



Fig. 4. Plasma flow around a monopolar j_z : The dashed lines are perturbation streamlines and the solid lines are the total flow streamlines.

and the associated z component of the magnetic perturbation is given by the requirement (16). Taking the linear form (16b) we have

$$\mathbf{b}_{\mathbf{Z}} = \mathbf{b}_{\boldsymbol{\omega}} \cos\varphi \, \tan\theta_{\mathbf{A}} \tag{21}$$

One may now compute the current system perpendicular to the characteristic by using (15).

$$(\mathbf{j}_{\mathbf{r}}, \mathbf{j}_{\varphi}) = \frac{1}{\mu_0} \nabla_{\!\!\wedge} \mathbf{b} \equiv \frac{1}{\mu_0} (\nabla \mathbf{b}_{\mathbf{z}})_{\wedge} \mathbf{\hat{z}}$$
 (22)

[cf. Neubauer 1980]. Evidently b_z is a stream function for the current flow in the (r, φ) plane and thus flow lines are described by $b_z = \text{const}$, i.e.,

$$r \cos \varphi \equiv x = const \quad r < r_0$$
 (23)

$$\frac{r^{0}}{r}\cos\varphi = const \quad r > r_{0} \quad (24)$$

For $r < r_0$ the current flow in the (r,φ) , or (x,y), plane, perpendicular to the characteristic, is in the y direction. The expression for streamlines of current flow outside $r = r_0$ is that describing a two-dimensional dipole that is aligned in the negative y direction. The current flow in the (x,y) plane is shown in Figure 2.

Outside the current-carrying cylinder aligned with the characteristic the current flow is just in the (x,y) plane. Inside $r = r_0$ we must superpose j_z to get the full three-dimensional current and thus the current flow inside the cylinder is in the (y,z) planes and is rectilinear.

Now using (15), we can express j_y in terms of j_z ;

$$\mathbf{j}_{\mathbf{y}} = -\frac{1}{2} \tan \theta_{\mathbf{A}} \mathbf{j}_{\mathbf{Z}0} \quad \mathbf{r} < \mathbf{r}_0 \tag{25}$$

The current inside the cylinder flows at an angle θ' to the direction of the background field, where

$$\sin \theta' = \frac{\sin \theta_{A}}{(4 + \tan^2 \theta_{A})^{\frac{1}{2}}}$$
(26)

The angle lies between the field direction and that of the axis to the cylinder. The angle it makes with the axis is θ^{m} :

$$\tan\theta'' = \frac{1}{2}\tan\theta_A \tag{27}$$

The total current flow in the cylinder is thus not confined to the cylinder. It must close outside in the exterior current system. The three-dimensional closure is sketched in Figure 3. The explicit form of j is

$$(j_X, j_V, j_Z) = j_{Z_0}(0, -\frac{1}{2}\tan\theta_A, 1)$$
 r0 (28a)

$$(j_X, j_Y, j_Z) = j_{Z_0}(r_0/r)^2 \tan \theta_A$$
.
 $(-\frac{XY}{r^2}, \frac{X^2 - Y^2}{2r^2}, 0)$ $r > r_0$ (28b)

The flow perturbation is proportional to the magnetic perturbation (6). Streamlines of the flow perturbation and the total velocity field are sketched in Figure 4. The nonlinear solutions for the field and current from (21) to (28) will have the same topology as the linear solution but will differ quantitatively. For example the dipolar perpendicular currents develop an asymmetry in the nonlinear solution.

4. Dipolar j_z : The Io Current System

The motion of a uniform conductor through a magnetized plasma will produce an Alfvénic disturbance with a dipolar distribution of j_z , to first order. Drell et al. [1965] considered the problem in the context of computing the drag experienced by an artificial satellite in the ionosphere. More recently Neubauer [1980] addressed the interaction of a natural satellite, Io, with the Jovian magnetoplasma (see also Goldreich and Lynden-Bell [1969], Goertz and Deift [1973], and Southwood et al. [1980]). The Voyager 1 magnetometer data has confirmed the existence of the dipolar current system (Acuna et al. [1981]).

Figure 5 illustrates the process. The plasma flow incident on the satellite will induce an electric field within Io (or in its immediate environment, e.g., any



Fig. 5. The production of Alfvén wings by Io. The magnetic field in the (x=0,y,z) plane is shown.



Fig. 6. The plasma flow perturbation (u) in a plane perpendicular to B_0 .

highly conducting atmosphere). The net electric field drives a current through the conductor that must close in the surrounding medium. the current is stationary in the Io frame of reference. It is carried off into the plasma as the characteristic-aligned current at the core of an Alfvénic structure. As the conductor is not a net emitter of current, the characteristic current system is bipolar. One expects current to flow through Io (or its atmsophere) and to be carried off, quite possibly in thin sheets (wings), from the flanks. Characteristic-aligned wings are set up both parallel, V^+ , and antiparallel, V^- , to the field. Now the strength of the Alfvénic disturbance that is generated can be measured by the strength of the characteristic-aligned current which in turn evidently depends on the nature of the currents and electric fields induced in Io or its environment. In the simplest models a simple conductance (or impedance) is attributed to Io, and the perturbation electric field is expressed as a fraction of the imposed electric field. The result depends on the ratio of satellite, \sum_{I} , to Alfvén conductance, \sum_{A} , in the external plasma [e.g., Neubauer, 1980; Southwood and Dunlop, 1984].

$$\mathbf{E}_{\mathbf{Io}} = -\mathbf{E}_{\mathbf{C}} \cdot \sum_{\mathbf{I}} / (2 \sum_{\mathbf{A}} + \sum_{\mathbf{I}})$$
(29)

where E_{IO} is the perturbation electric field at Io. Treatments such as that of Neubauer [1980] have put all the current in sheets on the edge of a cylinder, r = R_I , attached to the source. In such circumstances the electric potential is a harmonic function both inside and outside the cylinder. If the characteristic-aligned surface current is sinusoidally distributed around the cylinder, solutions of (11) take the simple form

$$\Phi = -E_{IO} R_I^2 \times / r^2 \qquad r > R_I$$

$$\Phi = E_{IO} \times \qquad r < R_I$$
(30)

Inside the cylinder of current the electric field is uniform and thus we may choose $E = E_{IO}$.

Contours of Φ yield perturbation flow lines. We illustrate these in Figure 6. From (6), one deduces that these are also perturbed field lines. The



Fig. 7. The direction of current flow on the surface of the Alfvén wing (for x > 0).

magnetic field perturbation is given to linear order in Cartesian coordinates by

$$(b_x, b_y, b_z) = b_0 (R_I^2 / r^4) (2xy \cos \theta_A,$$

 $(y^2 - x^2) \cos \theta_A, (y^2 - x^2) \sin \theta_A) r > R_I (31a)$

$$(b_x, b_y, b_z) = b_0(0, \cos\theta_A, \sin\theta_A) r < R_I$$
 (31b)

where the amplitude of the magnetic field perturbation is given by $b_0 = E_{IO}/V_A$. We shall continue with the linear solution to investigate the character of the fields and currents, as in the previous section.

The sheet current flow on the cylinder is interesting. The characteristic-aligned surface current is

$$\mathbf{j}_{\mathbf{ZS}} = -\frac{2\mathbf{b}_0}{\mu_0} \cos\theta_{\mathbf{A}} \cos\varphi \qquad (32)$$

and there is also an azimuthal component

$$\mathbf{j}_{\varphi \mathbf{S}} = \frac{2\mathbf{b}_0}{\mu_0} \sin\theta_\mathbf{A} \cos^2\varphi \tag{33}$$

Current streamlines in the sheet obey (from (32) and (33))

$$dz/R_{I}d\varphi = j_{zs}/j_{\varphi s} = -\frac{\cot\theta_{A}}{\cos\varphi}$$
(34)

Note how only at $\varphi = 0^{\circ}$ is the current flow strictly field aligned. One may integrate to find

$$z - z_0 = -R_I \cot \theta_A \cdot \ln[\tan(\pi/4 + \varphi/2)] \quad (35)$$

where z_0 is an integration constant. Figure 7 illustrates a streamline on the half cylinder, $-\pi/2 < \varphi < \pi/2$ (x>0).

The azimuthal surface current is not solenoidal and closes by dint of currents in the region, $r > R_{I}$. These may be derived by differentiation from equation (31). In Cartesian coordinates one has

$$j = (b_0 R_I^2 \sin \theta_A / \mu_0 r^6) \cdot (6x^2 y - 2y^3, 6xy^2 - 2x^3, 0) r > R_I$$
(36a)

$$\mathbf{j} = \mathbf{0} \mathbf{r} < \mathbf{R}_{\mathbf{I}}$$
(36b)

The expression for (j_x, j_y) when $r > R_I$ is that of a two-dimensional quadrupole. Furthermore, streamlines of (j_x, j_y) are along contours of b_z (from (22)). Taking the expression for b_z given in (31), streamlines of current flow $(r > R_I)$ are given by

$$r^{2}/R_{I}^{2} = c_{0}(\sin^{2}\varphi - \cos^{2}\varphi) = -c_{0}\cos(2\varphi) \quad (37)$$

where c_0 is a constant. Figure 8 shows the family of curves described by (37). The existence of current lobes at the front and back (along the y axis) has been mentioned by Southwood and Dunlop [1984] and Rasmussen et al. [1985]. They pointed out that the j_AB force in these lobes will slow the plasma down as it approaches the wing, and accelerate it as it recedes from the wing. The existence of the second pair of current lobes seems to have been missed to date.

The total current system closure is obtained by considering both volume and surface currents, i.e., superposing Figures 7 and 8. The resulting current system is sketched in Figure 9. It is interesting to note that in the central part of the surface current flow in Figures 7 and 9 where it is largest it is parallel to B_0 . In constructing Figure 9 we made the assumption that the surface currents were not actually sheet currents but a highly localized volume current, a point to which we return. The essential points to note here are that the lobes upstream and downstream of the obstacle are connected by current flow along the characteristic and in the current systems on the flanks there is a cascade of current which does not connect to the lobes up stream or downstream or on the other flank.

The magnetic force on the plasma (j_AB) will also be of a quadrupole-like form, but as it lies in a plane perpendicular to the field, its component in the flow direction is distorted by a multiplicative factor, $\sec \theta_A$. Figure 10 illustrates the direction of the force outside the cylinder, $r = R_I$. The lobe structure means that the radial force component changes direction at $\varphi = \pm 45^\circ$, the x component at $\varphi =$ $\pm 30^\circ$ and the component parallel to V_c at $\varphi \pm 60^\circ$ and 0° .

Forces can also be described by the magnetic tension force, represented by the first term on the right hand side of the momentum equation (2). The wave structure bends the field, and the tension force is simply visualized. Using the expressions for **b** in equation (31) we can derive the projection of a magnetic field line (in the region $r > R_{I}$, x > 0) onto the (y,z) plane and the plane defined by the x axis and the background field. Figure 11 shows the result. As one moves down the field, the field line drapes over the cylinder, and from the direction of



Fig. 8. The volume currents (j_x, j_y) around lo's Alfvén wings.

the bending one can picture the forces that act at various azimuths. the plasma in the region $60^{\circ} < \varphi$ < 90° is slowed down and pushed around the wing. For $30^{\circ} < \varphi < 60^{\circ}$, plasma is accelerated past the side and pushed around the wing, and for $0^{\circ} < \varphi < 30^{\circ}$ the plasma is accelerated past and toward the wing. The forces on the plasma in the other quadrants can easily be deduced from inspection of Figures 10 and 11. For purposes of deriving the figure, in particular the lengths marked on the field lines in Figure 11, we have used zeroth-order estimates. (The exact or first-order solutions are not The quantity x_0 is the x coordinate at analytic.) The full nonlinear solution can also be large z. constructed by using (16a) instead of (16b) when deriving (31) if desired. This merely distorts the solution presented here when nonlinearity is important; however, the solutions are qualitatively the same.

In the above treatment we have followed previous workers and assumed that the current flowing out of Io emerges in thin sheets. In constrast, in the preceding section we used a distributed characteristic current corresponding to the emission of current over the body of the source. In fact this is a more realistic assumption in the case of Io. Even were Io a highly conducting uniform sphere, there would be normal current flow over virtually the whole surface (compare the analogous electrostatic problem of the induced charge distribution on a uniform sphere immersed in an external field). In general there will be characteristic—aligned currents distributed over the cross—section of the cylinder. For example, consider

$$j_z = 2I(r^2/R_I^4)\cos\varphi \qquad r < R_I \qquad (38)$$

Such a distribution of current in the z direction is bipolar, the total current in each direction being I.

The sinusoidal azimuthal dependence determines that the solution outside the region carrying current in



Fig. 9. The closure of total current in the Alfven wing of Io. The front and back current lobes close with one another via the surface current. The side lobes (of which only the nearside ones are shown) form a cascade along the length of the wing.



Fig. 10. The direction of the force experienced by the plasma in the plane perpendicular to B_0 (x>0).

the z direction is exactly as described above. However, inside the cylinder the solution of (11) becomes

$$\Phi - \frac{-2}{15} \mu_0 V^{\pm} I \cos \varphi \ (r/R_I)^4 \tag{39}$$

One may compute fields and currents just as before. In this case in the center of the cylinder the flow and field lines are not rectilinear. The field and flow perturbations maximize in the center and, as there are no sheet currents, are continuous throughout the interior and exterior of the cylinder.

Notwithstanding the above remarks it is important to note that the exterior field, current and flow patterns are classifiable solely by the polarity of the characteristic current distribution. In the next section we build on this fact.

5. General Distribution of j_z

We have considered thus far two simple examples whose far field solutions correspond to a monopolar and dipolar characteristic current flow. These are the lowest-order solutions for two ideal cases, the first where the source is a net current emitter and the second where current is induced externally in a conductor by an imposed flow. In this section a mathematical formalism is developed that will be applicable to any distribution of j_z .

Invariance in z and the solenoidal nature of the magnetic field means that

$$\frac{\partial \mathbf{b}_{\mathbf{x}}}{\partial \mathbf{z}} = \frac{\partial \mathbf{b}_{\mathbf{x}}}{\partial \mathbf{x}} + \frac{\partial \mathbf{b}_{\mathbf{y}}}{\partial \mathbf{y}} = 0$$
(40)

One may thus write b_x and b_y in terms of a stream function, $\psi(x,y)$

$$(\mathbf{b}_{\mathbf{x}},\mathbf{b}_{\mathbf{v}}) - \nabla \psi_{\Lambda} \mathbf{\hat{z}}$$
(41)

Imposing the condition of constant magnetic field pressure gives

$$b_{z} = \left\{ B_{0}^{2} - \left[\frac{\partial \psi}{\partial y} \right]^{2} - \left[\frac{\partial \psi}{\partial x} + B_{0} \sin \theta_{A} \right]^{2} \right\}^{\frac{1}{2}} - B_{0} \cos \theta_{A}$$
(42a)

or

$$b_z = -tan\theta_A. \left[\frac{\partial \psi}{\partial x}\right]$$
 (42b)



Fig. 11. The projection of a field line in the region $r > R_I$, x > 0 onto the (a) (\hat{y}, \hat{z}) and (b) (\hat{x}, B_0) planes. The direction of the "field line tension" force is indicated for the different regions of space.

in the linear limit. θ_A is the angle between B_0 and the characteristic as before. In a similar manner the independent solenoidal nature of the current flow parallel and perpendicular to the characteristic yields

$$(\mathbf{j}_{\mathbf{X}},\mathbf{j}_{\mathbf{Y}}) = \nabla \Psi_{\Lambda} \mathbf{\hat{z}}$$
(43)

Now the two stream functions are related to one another by the x, y component of Ampere's law, (15), and the constant field pressure requirement, (16),

$$\Psi = b_Z/\mu_0 \tag{44a}$$

or

$$\Psi = -\frac{\tan\theta_{\rm A}}{\mu_0} \cdot \left[\frac{\partial\psi}{\partial x}\right] \tag{44b}$$

in linear limit. The gradient of any stream function is also a stream function whose source is twice the original polar order and orientated along the gradient. Thus if the characteristic current has a monopolar form then the (x,y) magnetic fields are those of a line current and the perpendicular currents have a dipolar form, just as we deduced in section 3. Similarly, the external solution in section 4 revealed a quadrupole perpendicular current associated with a dipolar z current and magnetic field pattern. As we have already mentioned, the nonlinear solutions will be qualitatively the same but will have asymmetries that are absent in the linear limit.

The stream function ψ is proportional to the electrostatic potential. Recalling (6), (10) and (13) one may show that

$$\psi = \Phi/V^{\pm} \tag{45}$$

Hence, given a j_z distribution a procedure for solving is to first calculate the electrostatic potential.

$$\Phi(\mathbf{r}) = \frac{\mu_0 V^{\pm}}{4\pi} \int \frac{j_Z(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 \mathbf{r}'$$
(46)

where $\mathbf{r} = (\mathbf{r}, \varphi)$. The stream functions in equations (44) and (45), ψ and Ψ , are easily calculable from Φ , and hence j and b are determined.

Just as in the dipole case in the previous section, if j_z is zero outside a finite region (say $r > r_0$), then outside r_0 the electrostatic potential (8) will satisfy a two-dimensional Laplace equation. The solutions to this equation in two dimensions are standard [e.g., Neubauer, 1980].

$$\Phi = A_{0}\Phi_{0} + A_{1}\Phi_{1} + \sum_{n=2}^{\infty} A_{n}\Phi_{n} + B_{n}\Phi_{n}$$
(47a)
$$\Phi = -A_{0}r\cos\varphi + A_{1}\ln(r_{0}/r) + \sum_{n=2}^{\infty} A_{n}e^{(A_{n}\cos\left[\frac{n\varphi}{2}\right]} + B_{n}\sin\left[\frac{n\varphi}{2}\right]) \cdot \left[\frac{r_{0}}{r}\right]^{n/2}$$
(47b)

The polar order of any given term is n.

The terms listed in (47) are orthogonal on the range corresponding to the volume outside the cylinder, so the coefficient of any term (Φ_n, Φ_n) will be

$$A_{n} = \int_{r>r_{0}}^{\Phi_{n}\Phi r dr d\varphi} \int_{r>r_{0}}^{\Phi_{n}^{2} r dr d\varphi}$$

$$B_{n} = \int_{r>r_{0}}^{\Phi_{n}^{2}\Phi r dr d\varphi} \int_{r>r_{0}}^{\Phi_{n}^{2} r dr d\varphi}$$
(48)

In this fashion it is possible to derive a series of coefficients that will also be the coefficients in a

similar expansion for the magnetic field. In turn the perpendicular current may be computed as being proportional to the same coefficients but with the polar order of each term appropriately raised when the linear approximation is applicable. In the nonlinear regime the current streamlines will be distortions of the linear solution, but topologically the same.

6. Discussion and Conclusions

We have examined general nonlinear solutions for a class of steady MHD flows that we have termed stationary Alfvénic structures. The magnetic and velocity fields and current distribution have been described and their relationships to one another discussed. The structures that we describe are important because they are a means of conducting a current through a plasma. Such currents can naturally arise when the plasma is in contact with a conductor, e.g., ionosphere-magnetosphere coupling or a conducting satellite (Io). Current is carried away from the sources by a characteristic-aligned current which is often loosely called the field-aligned current. (The magnetic field and characteristic directions coincide when there is no background flow.) The characteristic current, and the corresponding magnetic field perturbation, also give rise to a subsidiary current flow that is perpendicular to the characteristic direction and self-closing. These currents serve to redirect the plasma flow exterior to the characteristic current-carrying region in such a way that the plasma velocity perturbation and the magnetic field perturbation satisfy a propagating Alfvén wave. It is possible to derive the complete wave structure form characteristic current and free steam the magnetoplasma quantities. This is illustrated by two worked examples. The first is a monopolar source and is of potential use when considering closure from net current emitters (e.g., Birkeland currents in magnetosphere-ionosphere coupling and the core of twisted flux transfer events). The second example has a dipolar characteristic current pattern, and no net current is emitted from the source in this case. The dipolar example is potentially relevant to the bulk motion of flux tubes (e.g., flux transfer events, substorms) and the waves produced by natural and artificial satellites. The monopole and dipole solutions are particularly worthy of attention since one or the other will be the far-field solution to most space plasma stationary Alfvénic structures. We have also given the solution for a general distribution of characteristic current.

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D. J. Southwood, and A. N. Wright, Physics Department, Imperial college, Prince Consort Road, London SW7 2BZ, England.

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