# THE TRANSMISSION OF ALFVEN WAVES THROUGH THE IO PLASMA TORUS

### Andrew N. Wright and Steven J. Schwartz

# Astronomy Unit, School of Mathematical Sciences, Queen Mary College University of London

Abstract. In this paper we study the nature of Alfvén wave propagation through the Io plasma torus. A one-dimensional model is used with uniform magnetic field and exponential density decrease to a constant value. The solution can be expanded near the center of the torus and far away from the torus to give propagating Alfvén waves. The time-averaged Poynting flux is independent of position and, in the near and far limits, is equivalent to the sum of Poynting fluxes carried by individual wave trains. In this fashion it is possible to calculate the fraction of energy that is transmitted or reflected by the change in Alfvén speed through the torus, for a given wave train near Io. The solution is sensitive to the distant (or asymptotic) Alfvén speed. A lower limit for this speed can be found from the density decrease alone. Using this value we find, in accord with previous work, that there is negligible wave reflection. A more realistic asymptotic Alfvén speed takes into account the increase in field strength along the Io flux tube. This larger Alfvén speed value yields a significant reflection coefficient, and the result is in good agreement with our previous numerical solution. Our results imply that Io's Alfvén waves may not propagate completely through the plasma torus, and thus WKB theory and ray tracing may not provide meaningful estimates of the energy transport.

# 1. Introduction

In 1964, Bigg [1964] reported that observations of Jovian decametric emissions (DAM) were influenced by the position of Io. Goldreich and Lynden-Bell [1969] presented the first model of the Io-Jupiter interaction, and introduced the important notion that Io is a good electrical conductor. They envisaged that the current flowing through Io would close in the Jovian ionosphere via currents at the side of the Io flux tube. In 1973, Goertz and Deift [1973] (and Deift and Goertz[1973]) suggested that the current would be carried down to the Jovian ionosphere via Alfvén waves, which would then be reflected back to Io. With the discovery of the Io plasma torus [Bridge et al., 1979; Broadfoot et al., 1979], and the associated slower wave propagation speed, it was realized that the Alfvén waves would not return to Io but would extend downstream of the satellite [Neubauer, 1980; Gurnett and Goertz, 1981]. Evidence of this wake was found in Pioneer 10 magnetometer data [Walker and Kivelson, 1981]. More recent work [Goertz, 1983] has modeled the evolution of field lines that encounter Io.

There has been considerable interest in the structure of the waves downstream from Io because they are thought to produce decametric radiation. The planetary

Copyright 1989 by the American Geophysical Union.

Paper number 88JA04070. 0148-0227/89/88JA-04070\$02.00 radio astronomy observations from Voyager 1 and 2 [Warwick et al., 1979a,b] show sets of nested arclike structures in frequency-time spectra. The bursts of DAM is probably related to the Alfvén wave pattern extending behind Io [Gurnett and Goertz, 1981]. However, this work assumed that the wave would propagate through the magnetosphere in the WKB fashion (i.e., the wave does not suffer any reflection within the medium) and then would be reflected efficiently from the Jovian ionosphere. Bagenal [1983] continued this approach and computed the trajectory of the wave by a ray-tracing calculation that used realistic field and density models.

Clearly, the nature of Alfvén wave propagation through the Jovian magnetosplasma is intimately related to the wave structure that exists downstream from Io. The models of Gurnett and Goertz [1981] and Bagenal [1983] rely on the WKB approximation. Some calculations [Gurnett and Goertz, 1981; Goertz, 1980] suggest the WKB limit is applicable and that reflections form the plasma torus may be neglected. A recent study [Wright, 1987] solved the MHD wave equation numerically and used a realistic magnetic field and density distribution. The results indicate that an Alfvén wave launched from the magnetic equator can interact strongly with the medium as it propagates through the inhomogeneities. For the wave expected to be produced by Io these calculations predicted that about 75% of the power would be reflected from the medium alone, implying strong violation of the WKB approximation. One factor contributing to the discrepancy with earlier predictions involves a comparison of scale lengths of the wave and of the magnetoplasma. The work of Gurnett and Goertz [1981] took the scale of the wave to be the size of Io. This is reasonable for the perpendicular scale of the wave, but the parallel wavelength could be 1 or 2 orders of magnitude greater than this [Wright, [1987]. The parallel scale of the wave is the more relevant to the propagation of Alfvén waves. Moreover, the length scale associated with the wave changes in an inhomogeneous medium due to the varying Alfvén speed. This more general Alfvén wave propagation problem has received analytic attention in the past [Goertz, 1980], which we reexamine here in more detail. Our qualitative results agree with those of Goertz [1980] when similar parameter values are used in both cases. However, the larger asymptotic Alfvén speed which mimics the dipolar variation in the magnetic field strength reproduces the behavior presented by Wright [1987]. Additionally, our mathematical results differ in detail from the earlier formulation. Our results indicate that Alfvén waves generated by Io do not propagate in a WKB manner when realistic Alfvén speed profiles are imposed. We also give convenient forms for the energy fluxes, along with transmission and reflection coefficients, based on our analytical results.

The problem of Alfvén waves propagating through an inhomogeneous medium has also received considerable interest from the solar physics community (Ferraro and Plumpton, [1958]; see Hollweg [1983] for a brief review).

We draw freely on this body of work, and in particular on Leroy [1983], who solved a wave equation similar to that employed here.

The paper is structured in the following fashion: section 2 describes the model and gives the relevant solution; section 3 evaluates the energy fluxes and their interpretation; section 4 discusses the application of our results before the conclusions in section 5. In the appendix we investigate other analytical limits of our solution.

# 2. The Model

To investigate Alfvén wave propagation through the torus density distribution we shall adopt a one-dimensional magnetoplasma. Quantities vary with the z coordinate, and are indpendent of x and y. The background magnetic field is uniform and parallel to the z axis. This model is the same as that used by Goertz [1980], where it is shown that the equation governing perpendicular velocity perturbations (v) is

$$\frac{\partial^2 v}{\partial t^2} - V_A^2(z) \quad \frac{\partial^2 v}{\partial z^2} \tag{1}$$

We shall model the torus density profile as an exponential decrease from the equatorial value  $\rho_0$  (at z = 0) to the value far away ( $z \rightarrow \infty$ ) of  $\rho_{\infty}$ .

$$\rho(z) = \rho_{\infty} + (\rho_0 - \rho_{\infty}) \cdot \exp[-2z/z_0]$$
(2)

The scale length for the density variation is  $z_0/2$ . This yields the variation of Alfvén speed along the field lines to be [Goertz, 1980]

$$\frac{1}{v_{A}^{2}(z)} - \frac{1}{v_{A\infty}^{2}} + \left[\frac{1}{v_{A0}^{2}} - \frac{1}{v_{A\infty}^{2}}\right] \cdot \exp[-2z/z_{0}]$$
$$- \frac{1}{v_{A\infty}^{2}} + \frac{1}{v_{A1}^{2}} \cdot \exp[-2z/z_{0}]$$
(3)

 $V_{A_0}$  is the equatorial Alfvén speed (at z=0) and  $V_{A\infty}$  is the asymptotic value of the Alfvén speed as  $z \rightarrow \infty$ .  $V_{A_1}$  is defined via equation (3). Our wave equation is identical in form to equation (16) of Leroy [1983]. Following Goertz [1980] we shall change variables from z to x,

$$x = (\omega z_0 / V_{A_1}) . exp[-z/z_0]$$
 (4)

(There appear to be some typographical errors in Goertz [1980], including an additional factor of  $V_{A_1}$  in the definition of x.) We shall only consider one Fourier frequency, and let all fields have a time dependence of  $e^{i\omega t}$ . Thus the velocity perturbation is given by  $v = ue^{i\omega t}$ ; u has real and imaginary parts  $(u_r + iu_i)$ . The wave equation (1) for v has now become Bessel's equation for u:

$$x^{2} \frac{d^{2}u}{dx^{2}} + x \frac{du}{dx} + (x^{2} - \nu^{2})u = 0$$
 (5)

The parameter v is purely imaginary and is given by

$$\nu^{2} = -\omega^{2} z_{0}^{2} / V_{A \infty}^{2} = -k_{\infty}^{2} z_{0}^{2} \qquad \nu = +i k_{\infty} z_{0} \qquad (6)$$

The general solution to Bessel's equation is a linear combination of the Bessel functions  $J_{\nu}(x)$ ,  $J_{-\nu}(x)$  and requires two boundary conditions. We shall adopt the usual boundary condition [Goertz, 1980], namely, that the wave at large distances from Io (i.e.,  $z \rightarrow \infty$ ,  $x \rightarrow 0$ ) is propagating in the + 2 direction, so that there is no source of Alfvénic energy other than at z=0.

#### Solution at Large z

We consider first the small argument expansion for the Bessel functions [Abramowitz and Stegun, 1972]

$$J_{\nu}(x) \approx (x/2)^{\nu}/\Gamma(1+\nu)$$
, as  $x \rightarrow 0$ ,  $z \rightarrow \infty$  (7)

We shall denote the function  $G(\nu) = 1/\Gamma(1+\nu) = G_{\Gamma} + iG_{i}$  in terms of its real and imaginary parts, and take the complex coefficient of  $J_{\nu}(x)$  to be  $A = A_{\Gamma} + iA_{i}$ . The velocity, v, is given by

$$v = u(x)e^{i\omega t} = AJ_{\nu}(x)e^{i\omega t}$$
  
 $\approx (AG)exp[i\Phi_{\infty}]$  (8)

where  $\Phi_{\infty} = \omega t - k_{\infty} z + \varphi_{\infty}$ , and  $\varphi_{\infty} = k_{\infty} z_0 \ln[\omega z_0/2 V_{A_1}]$ . This corresponds to a forward traveling wave. A similar expression can be found for the  $J_{-\nu}(x)$  solution, which has a phase  $\Phi_{\infty} = \omega t + k_{\infty} z - \varphi_{\infty}$ . This describes a backward propagating wave, hence our boundary condition at large z requires the coefficient of  $J_{-\nu}$  to vanish [Goertz, 1980; Leroy, 1983]. Thus only  $J_{+\nu}$  contributes to our solution. We shall assume that the real part of v (i.e.,  $v_{\rm T}$ ) describes the physical velocity perturbation,

$$v_r \approx (A_r G_r - A_i G_i) \cos(\Phi_{\infty}) - (A_i G_r + A_r G_i) \sin(\Phi_{\infty})$$
(9)

### Solution at Smaller z

Near Io (at small z) the WKB limit is expected to be valid [Goertz 1980; Wright, 1987]. This requires that the scale length of the medium be much larger than one wavelength, i.e.,  $\omega z_0 / V_A >> 1$ . The variable x may be written in the following forms (using equations (3), (4) and (6)),

$$x = \frac{\omega z_0}{V_A} \left(1 - V_A^2 / V_{A\infty}^2\right)^{\frac{1}{2}} = i \nu I \left(V_{A\infty}^2 / V_A^2 - 1\right)^{\frac{1}{2}}$$
(10)

From the first expression in (10) it is evident that the large x limit is likely to satisfy the WKB criterion given above. To investigate this solution we shall use the expansion given in Abramowitz and Stegun [1972], which is valid as  $x \to \infty$  at fixed  $\nu$ . The latter expression in (10) shows this condition is achieved as  $V_A(z)/V_{A\infty} \to 0$ , and yields

$$J_{\nu}(x) \approx \cos(x - \pi/4 - \nu \pi/2) \cdot (2/\pi x)^{\frac{1}{2}}, \text{ as } x \rightarrow \infty$$
(11)

Suppose that at some z, say z', x(z') is sufficiently large that (11) is a good approximation. Then x(z) can be written as a Taylor series about z' (using equations (4) and (10)) as x(z) =  $\varphi_0 - \omega z/V_A(z')$ , where  $\varphi_0 = \omega(z_0+z')/V_A(z')$  and terms in  $(z-z')^2/z_0^2$  and  $V_A(z')/V_{A\infty}$ have been neglected. By using this form for x in the cosine term of (11), defining  $\Phi_0 = \varphi_0 - \pi/4$ ,  $k' = \omega/V_A(z')$ and K =  $k_\infty z_0 \pi/2$ , the complex velocity field (v = AJ<sub>v</sub>e<sup>iωt</sup>) can be written as

$$v \approx A \left[\frac{2 V_{A1}}{\pi \omega z_0}\right]^{\frac{1}{2}} e^{Z/2Z_0} \cdot \left[\cos(\Phi_0 - zk')\cosh K\right] (12) + i \sin(\Phi_0 - zk')\sinh K e^{i\omega t}$$

and the real velocity component as

$$v_{r} \approx \frac{1}{2} \left[ \frac{2V_{A1}}{\pi \omega z_{0}} \right]^{\frac{1}{2}} e^{Z/2Z_{0}} \cdot \left[ A_{r} \cos(\omega t - zk' + \Phi_{0}) e^{K} - A_{i} \sin(\omega t - zk' + \Phi_{0}) e^{K} + A_{r} \cos(\omega t + zk' - \Phi_{0}) e^{-K} - A_{i} \sin(\omega t + zk' - \Phi_{0}) e^{-K} \right]$$
(13)

The first two terms in the square brackets represent forward propagating waves, and the last two terms backward propagating waves. The propagation speed is the local Alfvén speed,  $V_A(z')$ . The exponential factor toward the beginning of equation (13) produces the well-known result of WKB theory that the amplitude of the velocity perturbation is inversely proportional to the fourth root of the plasma density. It is reasonable to expect this equation to describe Alfvén wave propagation near Io, since a typical angular frequency for Io's waves is  $10^{-1} \text{ s}^{-1}$  [Goertz, 1980; Wright,[1987],  $z_0 \sim 1 \text{ R}_J$  (1  $\text{R}_J = 7.14 \times 10^4 \text{ km}$ ) and  $V_A(0) \sim 400 \text{ km s}^{-1}$ . This yields a value of  $\omega z_0/V_A \sim 20$ , and means that the wavelength of the torus. The computations presented in Wright [1987] also show that the waves near Io propagate according to the WKB limit.

#### 3. Energy Fluxes and Interpretation

The steady state one-dimensional dissipation-free problem solved in the preceding section possesses an invariant quantity which is identified as the energy flux (see Leroy [1985] for a more complete discussion of this topic). For linear Alfvénic fluctuations the z component of the energy flux is simply the Poynting flux. In order to calculate this we must evaluate the magnetic field perturbation  $(b(z)e^{i\omega t})$  in terms of the velocity field  $(u(z)e^{i\omega t})$ , which can be done using Faraday's law,

$$i\omega b(z) = B_0 \frac{\partial u(z)}{\partial z}$$
  
b =  $i(B_0 A/V_{A_1}) \cdot e^{-z/z_0} \cdot J_p(x)$  (14)

(The prime denotes differentiation with respect to x.) The z component of the time-averaged Poynting flux is

Re[] means the real part of the argument, and the asterisk denotes the complex conjugate. The latter expression in (15) may be written as

$$\langle S_{Z}(z) \rangle = -\frac{1}{4} (u.b^{*} + u^{*}.b).B_{0}/\mu_{0}$$
 (16)

On substituting  $u = AJ_{\mu}$  and b from equation (14), this becomes

$$\langle S_{Z}(z) \rangle = \frac{-iB_{0}^{2}|A|^{2}e^{-Z/Z_{0}}}{4\mu_{0}V_{A1}} \cdot (J_{-\nu}J_{\nu}^{-}-J_{\nu}J_{-\nu}^{-}) \quad (17)$$

The bracketed terms are the Wronskian of  $J_{-\nu}$  and  $J_{\nu}$ . The Wronskian may also be written as  $W[J_{-\nu}, J_{\nu}] = (2i/\pi x).\sinh(k_{\infty}z_0\pi)$  using result 9.1.15 from Abramowitz and Stegun [1972]. Thus the time-averaged Poynting flux is

$$\langle S_{Z}(z) \rangle = \sinh(k_{\omega}z_{0}\pi) \cdot B_{0}^{2} |A|^{2} / (2\pi\omega z_{0}\mu_{0})$$
 (18)

which is indeed the desired invariant, since it is independent of z. Note that this expression is exact, i.e., it uses the full, unapproximated solution.

At large z the velocity perturbation is given by the expression in (9) and the Poynting flux (18) is carried solely by a forward propagating wave. At smaller z (if the waves obey the WKB limit) there will be a superposition of forward and backward propagating waves. In this region the energy flux (18) is the difference between that carried by the forward propagating wave and the energy flux carried by the backward propagating wave and the energy flux carried by the backward propagating wave and the energy flux carried by the backward propagating wave and the energy flux carried by the backward propagating wave at z = z' to be the Alfvén wave launched from Io, the backward propagating wave at z = z' to be the part of Io's Alfvén wave reflected from the inhomogeneity, and the wave at  $z \to \infty$  to be the part of Io's Alfvén wave that is transmitted through the torus.

To verify this interpretation, recall that the Poynting flux carried by an individual linear Alfvén wave propagating in a WKB medium is

$$S_{z} = Re[ue^{i\omega t}].Re[be^{i\omega t}].B_{0}/\mu_{0} = \rho v_{r}^{2} V_{A}$$
(19)

Thus the time-averaged Poynting fluxes carried by the transmitted wave at large z, and the forward and backward propagating waves at  $z = z^{-}$  are, respectively,

$$\langle S_z(z \rightarrow \infty) \rangle_t = \rho_\infty V_{A\infty} \langle V_t^2 \rangle_t$$
 (20a)

$$\langle S_{z}(z') \rangle_{f} = \rho(z') V_{A}(z') \langle V_{f}^{2} \rangle_{f}$$
 (20b)

$$\langle S_{z}(z') \rangle_{b} = \rho(z') V_{A}(z') \langle V_{T}^{2} \rangle_{b}$$
 (20c)

It is easy to show from (9) and (13) that

$$\langle V_{f}^{2} \rangle_{t} = \frac{1}{2} |A|^{2} |G|^{2}$$
 (21a)

$$\langle V_r^2 \rangle_f = e^{2K} |A|^2 / (4\pi x')$$
 (21b)

$$\langle V_{\Gamma}^{2} \rangle_{b} = e^{-2K} |A|^{2} / (4\pi x')$$
 (21c)

where x' = x(z'). Substitution of (21a) into (20a) can be shown to give the same Poynting flux as required by (18) (N.B.  $|G|^2 = \sinh(k_{\alpha}z_0\pi)/(k_{\alpha}z_0\pi)$  [Abramowitz and Stegun, 1972]). The difference of  $\langle S \rangle_f$  and  $\langle S \rangle_b$  can also be shown to be equivalent to (18) (except for the higher order terms neglected in the expansions of  $J_{\nu}$  and x to obtain (13)). This confirms the notion that the Poynting fluxes in (20) can be thought of in terms of incident, reflected and transmitted waves upon the torus satisfying

$$\langle S \rangle_{f} - \langle S \rangle_{b} = \langle S \rangle_{t}$$
 (22)

By viewing the solution in this fashion it is possible to calculate the energy transmission and reflection coefficients

$$C_{T} = \langle S_{Z} \rangle_{t} / \langle S_{Z} \rangle_{f} = 1 - \exp[-2\pi\omega z_{0}/V_{A\infty}]$$

$$C_{R} = \langle S_{Z} \rangle_{b} / \langle S_{Z} \rangle_{f} = \exp[-2\pi\omega z_{0}/V_{A\infty}]$$
(23)

As one would expect, the transmission coefficient increases toward unity with increasing wave frequency.

### 4. Discussion

The expression for the transmission coefficient in (23) facilitates a brief quantitative discussion. For example, using the same parameter values found by Goertz [1980] ( $\omega = 0.1 \text{ rad s}^{-1}$ ,  $z_0 = 2 \text{ R}_J$  and  $V_{A\infty} = 10^4 \text{ km s}^{-1}$ ) yields a reflection coefficient of  $1.3 \times 10^{-4}$ . This suggests that the wave power is almost entirely transmitted through the torus, as one would expect if the WKB limit were valid everywhere. We shall see below, however, that inclusion of the additional variation in Alfvén speed expected in a dipolar field gives rise to a qualitative and quantitative difference.

The study done by Wright [1987] was numerical and consisted of launching a variety of pulses through a realistic field geometry and density profile. The density distribution used in this work was based upon the Voyager 1 plasma intrument [Bagenal et al., 1985; F. Bagenal, private communication, 1986]. Wright [1987] produced a plot showing how the fraction of incident Poynting flux that was transmitted through the torus varied with wave period. We can construct a similar graph using equation (23) and compare it with that derived by Wright [1987]. However, we should not expect the two transmission coefficients to be identical, since the model in Wright [1987] used a dipole field geometry, had a Gaussian density distribution and considered an isolated Alfvén wave packet. The present study has a uniform magnetic field, exponential density profile, and a harmonic time dependence. Nevertheless, the two studies should show a similar behavior.

In order to compare the transmission coefficient given in (23) with the results of Wright [1987], it is necessary to choose suitable numerical values for the basic parameters at some  $z_{min}$  and  $z_{max}$ . For  $z_{min}$  we take  $B_0 \approx 1900$  nT and a plasma density of  $\rho \approx 5 \times 10^4$ amu/cm<sup>3</sup> (e.g., Tokar et al., [1982], assuming a mean ionic mass of 25 amu). At  $z_{max} \approx 5 R_J$  (i.e., at a latitude of 45° on the same field line) the dipolar field strength is  $B \approx 3.6 \times 10^4$  nT, while  $\rho \approx 50$  amu/cm<sup>3</sup>. These values give corresponding Alfvén speeds of  $V_A(z_{min}) \approx 200$  km s<sup>-1</sup>,  $V_A(z_{max}) \approx 10^5$  km s<sup>-1</sup>. Therefore the effecive scale length for the variation in the Alfvén speed is found crudely from

$$V_{A}(z_{max})/V_{A}(z_{min}) \approx \exp[(z_{max}-z_{min})/z_{0}]$$
(24)

yielding  $z_0 \approx 0.8$  R<sub>J</sub>. Using only the density profile of (2) and ignoring the field variation would give a value of  $z_0 \approx 2.5$ . Although this higher value of  $z_0$  is the one most closely related to the theoretical, uniform magnetic field analysis of this paper, it is probably the variation in wave speed which is more relevant to the reflection process. Accordingly, we shall adopt a value of  $z_0 = 1$  R<sub>J</sub> for the purposes of numerical examples. This value also corresponds to the minimum density scale length in the model of Wright [1987], therby facilitating a comparison between those results and the present work. Inspection of (23) reveals that an increase by a factor of 2, say, in  $z_0$  would square the reflection coefficient.

Figure 1 is a plot of the transmission coefficient against T for the two models. The solid line represents the results of Wright [1987], and the T axis corresponds to the duration of the Alfvén wave packet launched form the equator. The dashed line shows the variation of the transmission coefficient given by equation (23), and in this case the T axis is interpreted as the time period of the wave. The typical value of T corresponding to Io's Alfvén waves is 0.34 in units of  $R_J/V_{Aeq}$  (= 179 s) giving a wave period of 60 s, i.e.,  $\omega \approx 0.1$  rad s<sup>-1</sup> as in previous work [Goertz, 1980].

The two curves in Figure 1 show qualitatively the same behavior and suggest that at least one half of the power in the wave launched from Io will be reflected by the plasma torus and field inhomogeneities. The difference between the two curves is probably due to the fact that Wright's [1987] model used a dipolar field geometry. The latter effect was neglected in the calculation presented in this paper. (It should be noted that we have



Fig. 1. The variation of the energy transmission coefficient ( $C_T$ ) with T. The solid line is the result of Wright [1987], and the dashed line is calculated using equation (23). T represents the duration of the pulse for the former study, and the time period of oscillation for the latter. The results show the same qualitative behavior. The difference in the two models can probably be attributed to the absence of a realistic field geometry in the present study. (See the text for more detailed discussion.) The period of waves that are likely to be produced by Io is around 0.34 R<sub>J</sub>/V<sub>Aeq</sub>, suggesting that not more than half of the power launched from Io will propagate, unimpeded, beyond a latitude of  $45^{\circ}$ .

chosen  $z_0$  and  $V_{A\infty}$  in order to compare our solution with results given by Wright [1987]. The latter study examined the propagation of waves through the torus to a latitude of 45°, and not propagation all the way to the Jovian ionosphere.)

Previous work [Goertz, 1980] has also studied the power carried by waves at z = 0 and  $z \rightarrow \infty$ , using a similar calculation to the one presented here, and interpreted the difference between Poynting fluxes at various positions in terms of the transmission and reflection properties of the medium. We have been unable to completely reconcile the detailed mathematical results given there with the present work, although some of the discrepancies are probably typographical. More importantly, although that work derives an expression for the Poynting flux (their equations 31,32) based on the total velocity perturbation (equivalent to our equations (15)-(17)). Equation (37) of that paper seems to imply that this flux is not independent of z. Perhaps we have misinterpreted the notation used by Goertz [1980].

# 5. Conclusions

In this paper we have studied how Alfvén waves propagate through a density distribution that is similar to that of the Io plasma torus. We are able to interpret our solution in terms of a wave that is incident upon the torus, a reflected wave, and a wave that is transmitted through the torus. The solution conforms to the steady state requirement that the net energy flux is independent of position. The results can be compared with those of Goertz [1980] and Wright [1987]. In the former model the Alfvén speed changes soley due to the density distribution. This yields a lower limit for the asymptotic Alfvén speed, since the magnetic field strength increases by over 2 orders of magnitude along the Io flux tube. Using this lower limit we find that the density change of the torus reflects a very small amount of wave power, in qualitative agreement with Goertz [1980], despite some mathematical discrepancies between the two solutions. The numerical results presented by Wright [1987] employed a realistic density variation and a dipolar background magnetic field. In order to compare our results with this study, we used an asymptotic Alfvén speed a factor of 7 times larger than the lower limit considered by Goertz [1980]. We find that a significant fraction of the Alfvén wave power launched from Io will not be transmitted through the torus and field variations, and that internal reflections from the Jovian magnetoplasma cannot be neglected. At least half of the power launched from Io will be reflected by inhomogeneities up to  $45^{\circ}$  latitude, and possibly even more will be reflected from the medium between  $45^{\circ}$  latitude and the Jovian ionosphere.

In reality the fate of the reflected component will probably be to traverse the torus many times before escaping to be dissipated in the ionosphere, or else it may be coupled to other wave modes via the medium's inhomogeneities (particularly those of the magnetic field).

# Appendix

In this appendix we show how it is possible to recover the WKB solution for the entire medium from our exact solution v=AJ<sub> $\nu$ </sub>(x)e<sup>i $\omega$ t</sup>. This is achieved by considering the large index,  $|\nu|$ , expansion described in §8.6 of Watson [1922]. (For the example of Io's Alfvén wave propagation  $|\nu| < 1$ , and so the solution given in this appendix is not applicable to that situation. However, there are many other examples where the solution may be useful, e.g., isothermal atmospheres under gravity, and we give it here for completeness.)

As  $|\nu| = \omega z_0 / V_{A\infty}$  becomes large we find that

$$v_{r}(z,t) \approx C. [\rho_{\infty}/\rho(z)]^{\frac{1}{4}} [A_{r}\cos(\omega t - \varphi_{wkb}) - A_{1}\sin(\omega t - \varphi_{wkb})]$$
(A1)

where  $C=\frac{1}{2}.e^{K}.[2V_{Acc}/(\pi\omega z_0)]^{\frac{1}{2}}$  and  $\varphi_{wkb}=\omega \int_{a}^{Z} dz^*/V_A(z^*)$ . This result can be found after some algebra and using the identity  $\arcsin(\alpha) = \int_{0}^{\infty} dt/(1+t^2)^{\frac{1}{2}}$ . The expression (A1) also agrees with the other forms derived for  $v_r$  (9),(13) when the approximations used in each calculation are borne in mind.

Acknowledgments. This work was carried out while A.N.W. was supported by a SERC Fellowship.

The Editor thanks M. H. Acutia and C. K. Goertz for their assistance in evaluating this paper.

## References

- Abramowitz, M., and I .A. Stegun, <u>Handbook of</u> <u>Mathematical Functions</u>, Dover, Publications Inc., New York, 1972.
- Bagenal, F., Alfvén wave propagation in the Io Flasma torus, <u>J. Geophys. Res.</u>, 88, 3013, 1983.
  Bagenal, F., R. L. McNutt, Jr., J. W. Belcher, H. S.
- Bagenal, F., R. L. McNutt, Jr., J. W. Belcher, H. S. Bridge, and J. D. Sullivan, Revised ion temperatures for Voyager plasma instruments in the Io plasma torus, <u>J. Geophys. Res.</u>, <u>90</u>, 1755, 1985.
  Bigg, E. K., Influence of the satellite Io on Jupiters
- Bigg, E. K., Influence of the satellite Io on Jupiters decametric emission, <u>Nature</u>, 203, 1008, 1964.
- Bridge, H. S., J. W. Belcher, A. J. Lazarus, J. D. Sullivan, R. L. McNut, F. Bagenal, J. D. Scudder, E. C. Sittler, G. L. Siscoe, V. M. Vasyliunas, C. K. Goertz, and C. M. Yeats, Plasma observations near Jupiter: Initial results from Voyager 1, <u>Science</u>, <u>204</u>, 987, 1979.
- Broadfoot, A. L., M. J. S. Belton, P. Z. Takacs, B. R. Sandal, D. E. Shemansky, J. B. Bertaux, J. E. Blamon, D. F. Stobel, J. C. McConnell, A. Dalgarno, R. Goody, and M. B. McElroy, Extreme ultraviolet observations from Voyager 1 encounter with Jupiter, Science, 204, 979, 1979.
- with Jupiter, <u>Science</u>, <u>204</u>, 979, 1979. Deift, P. A., and C. K. Goertz, The propagation of Alfvén waves along Io's flux tube, <u>Planet. Space</u> <u>Sci.</u>, <u>21</u>, 1417, 1973.
- <u>Sci.</u>, 21, 1417, 1973. Ferraro, V. C. A. and C. Plumpton, <u>Astrophys. J.</u>, <u>129</u>, 459, 1958.
- Goertz, C. K. Io's interaction with the plasma torus, <u>J.</u> <u>Geophys. Res.</u>, <u>85</u>, 2949, 1980.
- Goertz, C. K., The lo-control of Jupiter's decametric radiation: The Alfvén wave model, <u>Adv. Space Res.</u>, <u>3</u>, 59, 1983.
- Goertz, C. K., and P. A. Deift, Io's interaction with the magnetosphere, <u>Planet. Space Sci.</u>, <u>21</u>, 1399, 1973.
- Goldreich, P., and D. Lynden-Bell, Io, a Jovian unipolar inductor, <u>Astrophys. J.</u>, <u>156</u>, 59, 1969.
- Gurnett, D. A., and C. K. Goertz, Multiple Alfvén wave reflections excited by Io: Origin of the Jovian decametric arcs, <u>J. Geophys. Res.</u>, <u>86</u>, 717, 1981.
- Hollweg, J. V., Coronal heating by waves, Proceedings of Solar Wind 5, edited by M. Neugebauer, <u>NASA</u> <u>Conf. Publ.</u>, <u>2280</u>, 1983.

- Leroy, B., Propagation of Alfvén waves in an isothermal atmosphere when the displacement current is not neglected, Astron. Asrophys., 125, 371, 1983.
- Neubauer, F. M., Nonlinear standing Alfvén wave current system at Io: Theory, J. Geophys. Res., 85, 1171, 1980.
- Tokar, R. L., D. A. Gurnett, F. Bagenal, and R. R. Shaw, Light ion concentrations in the Jupiter's inner magnetosphere, <u>J. Geophys. Res.</u>, <u>87</u>, 2241, 1982.
- Walker, R., and M. G. Kivelson, Multiply reflected standing Alfvén waves in the Io torus: Pioneer 10
- standing Aliven waves in the to torus; Fioneer to observations, <u>Geophys. Res. Lett.</u>, 8, 1281, 1981.
  Warwick, J. W., J. B. Pearce, A. C. Riddle, J. K. Alexander, M. D. Desch, M. L. Kaiser, J. R. Thiemann, T. D. Car, S. Gulkis, A. Boischot, C. C. Harvey, and B. M. Pederson, Voyager 1 planetary radio astronomy observations near Jupiter, Science, 204, 995, 1979a.
- Warwick, J. W., J. B. Pearce, A. C. Riddle, J. K. Alexander, M. D. Desch, M. L. Kaiser, J. T. Thieman, T. D. Carr, S. Gulkis, A. Boischot, Y.

Leblanc, B. M. Pederson, and D. H. Staelin, Planetary radio astronomy observations from Voyager 2 near Jupiter, <u>Science</u>, <u>206</u>, 991, 1979b.

- Watson, G. N., A treatise on the Theory of Bessel Functions, Cambridge University Press, New York, 1922.
- Wright, A. N., The interaction of Io's Alfvén waves with the Jovian magnetosphere, J. Geophys. Res., 92, 9963, 1987.

S. J. Schwartz and A. N. Wright, Astronomy Unit, School of Mathematical Sciences, Queen Mary College, University of London, Mile End Road, London E1 4NS, England.

> (Received May 24,1988; revised November 7, 1988; accepted November 7, 1988.)