Nonstationary driven oscillations of a magnetic cavity

M. S. Ruderman[a] and A. N. Wright[b]
School of Mathematics and Statistics, University of St. Andrews, St. Andrews, Fife KY16 9SS, Scotland, United Kingdom

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The problem of transition to the steady state of driven oscillations in a magnetic cavity in a cold resistive plasma is addressed. The foot point driving polarized in the inhomogeneous direction is considered, and it is assumed that the cavity length in the direction of the equilibrium magnetic field is much larger than the cavity width in the inhomogeneous direction. The latter assumption enables one to neglect the variation of the magnetic pressure in the inhomogeneous direction, which strongly simplifies the analysis. The explicit solution describing the nonstationary behavior of the magnetic pressure and the velocity is obtained. This solution is used to study the properties of the transition to the steady state of oscillation. The main conclusion is that, in general, there are two different characteristic transitional times. The first time is inversely proportional to the decrement of the global mode. It characterizes the transition to the steady state of the global motion, which is the coherent oscillation of the cavity in the inhomogeneous direction. The second time is the largest of the two times, the first transitional time and the phase-mixing time, which is proportional to the magnetic Reynolds number in $\frac{1}{3}$ power. It characterizes the transition to the steady state of the local motion, which is oscillations at the local Alfvén frequencies, and the saturation of the energy damping rate. An example from solar physics shows that, in applications, the second transitional time can be much larger than the first one. © *2000 American Institute of Physics.*
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I. INTRODUCTION

Magnetic cavities are common in solar and space physics. Well-known examples of magnetic cavities are the magnetospheric cavity and the solar coronal loops. Studying oscillations in magnetic cavities is important for explaining such phenomena as excitation of ultra-low-frequency (ULF) waves in the magnetosphere and solar coronal heating.

There are two scenarios of excitation of oscillations in magnetic cavities. The first scenario is the so-called lateral driving, where oscillations are excited by an incoming fast magnetosonic wave. The second scenario is the foot point driving. In this scenario oscillations are excited by driving the magnetic field lines at one end of the cavity. In this article we consider only the second scenario; however, from the mathematical point of view, there is not very much difference between the two.

When the cavity is inhomogeneous in at least one spatial direction perpendicular to the equilibrium magnetic field, the eigenfrequencies of oscillations of different magnetic field lines are different. These eigenfrequencies form the Alfvén continuum. More precisely, there is a countable set of Alfvén continua, the first of them formed by the fundamental frequencies of the magnetic field lines, and the other by the overtones. If such an inhomogeneous cavity is driven harmonically, it is possible that the driving frequency matches the local Alfvén frequency at some position. This position is called the resonant position.

Let us consider a cavity inhomogeneous only in one spatial direction in planar geometry, or in the radial direction in cylindrical geometry. Let it be driven in the direction perpendicular both to the inhomogeneity direction and to the equilibrium magnetic field, by a driver that is independent of this direction, in planar geometry, or in the azimuthal direction by the axisymmetric driver in cylindrical geometry. Such a driving excites purely Alfvén oscillations. Initially, the motion of each magnetic field line is a superposition of two oscillations: one with the driving frequency, and the other with the local Alfvén frequency. Since the local Alfvén frequency varies in space oscillations of the neighboring magnetic field lines become more and more out of phase. This process is called phase mixing.[1] It causes the buildup of large gradients, which leads to the efficient damping of the oscillations with the local Alfvén frequencies on the phase-mixing time-scale proportional to $R^{1/3}$. Here R is either the viscous Reynolds number, or the magnetic Reynolds number, or the total Reynolds number in plasmas where both viscosity and resistivity operate. After this transitional phase-mixing time the cavity attains the steady state of driven oscillation, where all magnetic field lines oscillate with the driving frequency. The amplitudes of these oscillations are of the order of the driving amplitude everywhere except a narrow dissipative layer embracing the resonant position. The thickness of this layer is proportional to $R^{-1/3}$, and the amplitude of oscillations in it to $R^{1/3}$. The analytical solution describing the steady state of driven Alfvén oscillation in a one-dimensional magnetic cavity has been given by Ruder-

[a]Present address: Department of Applied Mathematics, University of Sheffield, Hicks Building, Western Bank, Sheffield S3 7RH, England. Electronic mail: michaelr@mcs.st-and.ac.uk
[b]Electronic mail: andy@mcs.st-and.ac.uk

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man *et al.*,[2] and the transition to the steady state of oscillation has been studied by, e.g., Ruderman.[3]

In what follows we consider only one-dimensional magnetic cavities in planar geometry. When such a cavity is driven in the inhomogeneous direction, this driving excites fast waves. If, in addition, the driving amplitude varies in the direction perpendicular both to the inhomogeneous direction and to the equilibrium magnetic field, the fast waves interact with Alfvén waves and excite the local Alfvén oscillations. This interaction is particularly strong near the resonant position. After the transitional time the dissipative layer embracing this resonant position is formed. Once again its thickness is proportional to $R^{-1/3}$, and the amplitude of oscillations in it to $R^{1/3}$. Now the motion in the dissipative layer is not purely Alfvénic. It is a superposition of fast and Alfvén waves, however, the Alfvénic component dominates. The solution describing this motion was given by, e.g., Kappraff and Tataronis,[4] Davila,[5] and Goossens *et al.*[6] (see also the review paper by Goossens and Ruderman[7]).

A very important property of magnetic cavities is the existence of so-called global modes. The global modes are the solutions to the eigenvalue problem for the linear dissipative magnetohydrodynamic (MHD) equations characterized by the property that the imaginary parts of the corresponding eigenvalues are much smaller than the real parts (the eigenvalue problem is obtained by taken perturbations of all quantities proportional to $e^{-i\omega t}$). The complex eigenfrequencies of the global modes tend to the limiting values with nonzero imaginary parts in the limit of vanishing dissipation. These global modes describe weakly damped free oscillations of magnetic cavities.

It was Kivelson and Southwood[8] who have first pointed out the importance of the global modes for magnetospheric physics. Since, in the magnetospheric cavity, different magnetic field lines oscillate with different frequencies, and sources of excitation of MHD waves in the magnetosphere have a broadband frequency spectrum, the resonant condition can be matched at an infinite number of geomagnetic field lines. Thus, every field line should be in resonance for a broad enough energy source. However, ground, ionospheric, and space observations indicate the existence of only one or a few resonant field line oscillations. Kivelson and Southwood[8] have suggested that a broadband source will first excite one or a few magnetospheric global modes (more precisely, the motion very close to that in a global mode except in a vicinity of a resonant position). These global modes then act as drivers exciting large-amplitude ULF waves at resonant positions. The global modes thus select the frequency of the observed ULF waves. After this pioneering work the global cavity modes remain very popular in magnetospheric physics. The selection of preferred frequencies by global modes has been demonstrated in numerical work for impulsive driving[9] and random driving.[10]

In solar physics the importance of global modes is mainly related to the problem of coronal heating. Since the width of dissipative layers embracing ideal resonant positions is proportional to $R^{1/3}$, in weakly dissipative plasmas the wave motion in dissipative layers is characterized by the presence of large spatial gradients, which causes strong wave energy dissipation and, as a consequence, efficient plasma heating. In the limit of vanishing dissipation the heating rate is independent of dissipative coefficients. Strongly enhanced wave energy dissipation in dissipative layers in weakly dissipative plasmas is called resonant absorption. The possibility of efficient heating by wave energy dissipation even in weakly dissipative plasmas has drawn considerable attention of plasma physicists to resonant absorption. Ionson[11] suggested resonant absorption as a possible mechanism of heating of coronal loops. Since then resonant absorption has remained a popular mechanism for explaining solar coronal heating (e.g., Refs. 12–16).

It turns out that the efficiency of resonant absorption strongly depends on the driving frequency. It is most efficient in the case of quasi-resonant driving, where the driving frequency is close to the frequency of one of the global modes (e.g., Refs. 17–20). This fact has attracted the attention of scientists studying resonant MHD waves to global modes (e.g., Refs. 21 and 22).

Resonant absorption is essentially a stationary process. However, nonstationary aspects in the theory of resonant MHD waves are also very important. In magnetospheric physics resonant waves are excited by the external driving with a finite duration in time. It is very important to know what is the maximum amplitude of the resonant oscillation and the minimum spatial scale that can be reached during this excitation. In solar physics the nonstationary behavior of MHD waves is interesting because coronal magnetic structures (e.g., coronal loops) normally exist only for a period of a few days or less. Resonant absorption can contribute into heating of such structures only if the transitional time after it starts to operate is shorter than the lifetime of these structures.

To the best of our knowledge the nonstationary behavior of resonant MHD waves was first addressed by Kappraff and Tataronis.[4] These authors studied the transition to the steady state of driven oscillation in the approximation of incompressible plasmas, and showed that the characteristic transitional time is the phase-mixing time proportional to $R^{1/3}$. After this time the dissipative layer is formed. Lee and Roberts[23] have studied the damping of a standing surface wave on a thin transitional layer in an incompressible ideal plasma. They have shown that the damping of the surface wave occurs because its energy is transferred into a thin energy-containing layer embracing the resonant position. In ideal plasmas the energy does not dissipate. It is stored in the energy-containing layer in the form of large-amplitude Alfvén waves. The transition to the steady state of driven oscillation in a magnetic cavity has then been studied numerically (e.g., Refs. 24 and 25).

In general, the transition to the steady state of driven oscillation in a magnetic cavity can be studied only numerically. However, there is one exception. Hollweg[26] has studied resonant absorption of MHD waves in a thin transition layer, i.e., under the assumption that the wavelength of the surface wave is much larger than the layer thickness. He pointed out that in this case it is possible to neglect the variation of the total pressure across the layer. This approximation enabled Hollweg to carry out the analysis for an ar-

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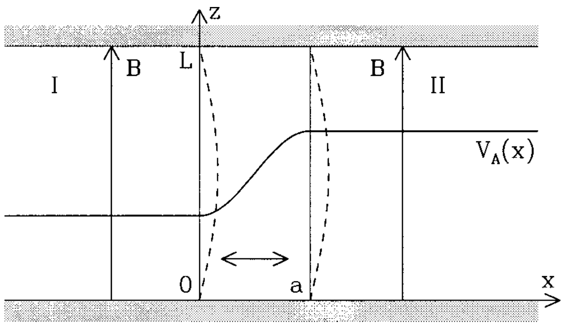

FIG. 1. The sketch of the equilibrium state. The dashed lines show the perturbed boundaries of the magnetic cavity. The horizontal double arrow shows the direction of the driving.

bitrary variation of the equilibrium density and magnetic field in the layer. Hollweg studied a standing wave, which, from the mathematical point of view, is completely equivalent to studying oscillations of a magnetic cavity.

In this article we study the transition to the steady state of driven oscillation in a thin magnetic cavity in a cold resistive plasma using Hollweg's approximation, i.e., neglecting the variation of the total pressure across the cavity. The paper is organized as follows. In the next section we formulate the problem, and derive equations that are used to study oscillations of the magnetic cavity. In Sec. III we obtain the solution describing the temporal evolution of the magnetic pressure and the velocity in the cavity. In Sec. IV the global modes of oscillation of the cavity are studied. In Sec. V the transition to the steady state of driven oscillation is investigated for the global motion of the cavity. In Sec. VI we address the nonstationary behavior of the velocity. In Sec. VII the energy dissipation rate is calculated. In Sec. VIII we give a summary and our conclusions. In particular, we included the table where the main results are collected.

II. DERIVATION OF GOVERNING EQUATIONS

We consider nonstationary oscillations of a magnetic cavity, driven at one of its ends, in a cold plasma (see Fig. 1). The equilibrium magnetic field B is constant and in the z direction of the Cartesian coordinates x, y, z. The equilibrium density ρ is a function of x. It is constant in region I determined by $x<0$, and in region II determined by $x>a$, and it only varies in the slab determined by $0<x<a$.

The plasma in the cavity is resistive; however, resistivity is assumed to be weak, so it is only important in regions with large spatial gradients. The plasma motion is governed by the system of linearized MHD equations

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{V_A^2}{B}\frac{\partial b_x}{\partial z}, \tag{1}$$

$$\frac{\partial v}{\partial t}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\frac{V_A^2}{B}\frac{\partial b_y}{\partial z}, \tag{2}$$

$$\frac{\partial b_x}{\partial t}=B\frac{\partial u}{\partial z}+\eta\nabla^2 b_x, \tag{3}$$

$$\frac{\partial b_y}{\partial t}=B\frac{\partial v}{\partial z}+\eta\nabla^2 b_y, \tag{4}$$

$$\frac{\partial P}{\partial t}=-\rho V_A^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\eta\nabla^2 P. \tag{5}$$

Here $\mathbf{v}=(u,v,0)$ is the velocity, $\mathbf{b}=(b_x,b_y,b_z)$ is the perturbation of the magnetic field, $P=Bb_z/\mu$ is the perturbation of the magnetic pressure, $V_A=B(\mu\rho)^{-1/2}$ is the Alfvén speed, η is the coefficient of magnetic field diffusion, and μ is the magnetic permeability of free space. In what follows we shall see that large spatial gradients are only present in the slab $0<x<a$, and only in the x direction. This observation enables us to neglect resistivity in regions I and II, and to write $\partial^2/\partial x^2$ instead of ∇^2.

The magnetic field lines are assumed to be frozen in the infinitely conducting plasmas at $z<0$ and $z>L$. The plasma in the region $z>L$ is immovable, while the plasma in the region $z<0$ moves with the velocity $f(t,x)$ in the x direction at $z=0$. As a result we have the boundary conditions

$$u=f(t,x),\quad v=0,\quad P=0 \qquad \text{at } z=0, \tag{6}$$

$$u=0,\quad v=0,\quad P=0 \qquad \text{at } z=L. \tag{7}$$

Note that the boundary conditions for P are not independent. They follow from the boundary conditions for u and v and Eq. (5) with $\partial^2/\partial x^2$ substituted for ∇^2. In what follows we assume that the characteristic scale of variation of f is much larger than a, so that we take $f(t,x)\approx f(t,0)$ for $0<x<a$.

We assume that $f=\partial f/\partial t=0$ and the system is at rest for $t\leqslant 0$, so perturbations of all quantities and their first derivatives with respect to time are zero at $t=0$.

Since the cavity is homogeneous in the y direction, we can Fourier analyze perturbations of all quantities and take them proportional to $\exp(iky)$. Then, eliminating b_x and b_y from Eqs. (1)–(4), we obtain

$$\left(\frac{\partial^2}{\partial t^2}-V_A^2\frac{\partial^2}{\partial z^2}\right)\frac{\partial u}{\partial t}-V_A^2\eta\frac{\partial^4 u}{\partial x^2\partial z^2}=-\frac{1}{\rho}\frac{\partial^3 P}{\partial t^2\partial x}, \tag{8}$$

$$\left(\frac{\partial^2}{\partial t^2}-V_A^2\frac{\partial^2}{\partial z^2}\right)\frac{\partial v}{\partial t}-V_A^2\eta\frac{\partial^4 v}{\partial x^2\partial z^2}=-\frac{ik}{\rho}\frac{\partial^2 P}{\partial t^2}. \tag{9}$$

Elimination of v from Eqs. (5) and (9), and the use of Eq. (8), yields

$$\left(\frac{\partial^2}{\partial t^2}-V_A^2\frac{\partial^2}{\partial z^2}\right)\frac{\partial^2 u}{\partial t\partial x}-V_A^2\eta\frac{\partial^4 u}{\partial x^3\partial z^2} =-\frac{1}{\rho V_A^2}\left(\frac{\partial^2}{\partial t^2}-V_A^2\frac{\partial^2}{\partial z^2}+V_A^2k^2\right)\frac{\partial^2 P}{\partial t^2}. \tag{10}$$

It follows from Eqs. (8) and (10) that u and P also satisfy the equation

$$\frac{\partial^2 P}{\partial t^2}-V_A^2\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial z^2}-k^2P\right)=V_A^2\frac{d\rho}{dx}\frac{\partial u}{\partial t}. \tag{11}$$

Equations (8)–(11) and the boundary conditions (6) and (7) will be used in what follows to study the driven oscillations of the magnetic cavity. Note that Eqs. (8)–(11) are not independent because there are four equations for three vari-

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ables. However, it turns out that Eq. (8) is suitable to determine u in regions I and II, while it is more convenient to use Eq. (10) to describe u in the inhomogeneous slab.

III. SOLUTION FOR THE MAGNETIC PRESSURE IN THE INHOMOGENEOUS LAYER

Since the quantities v and P are zero at $z=0,L$ they can be expanded into the Fourier series

$$v=\sum_{n=1}^{\infty} v_n(t,x)\sin(nlz), \quad P=\sum_{n=1}^{\infty} P_n(t,x)\sin(nlz), \tag{12}$$

where $l=\pi/L$. To use a similar expansion for u we make the variable substitution

$$u=U+\left(1-\frac{z}{L}\right)f(t,x). \tag{13}$$

Then the quantity U is zero at $z=0,L$ and can be expanded into a Fourier series similar to Eq. (12). The equations for U are obtained from Eqs. (8) and (10) by substituting U for u, and adding the terms $(z/L-1)\partial^3 f/\partial t^3$ and $(z/L-1)\partial^4 f/\partial t^3\partial x$ to the right-hand sides of Eqs. (8) and (10), respectively.

In what follows the values of $\partial P_n/\partial x$ calculated at the boundaries of the inhomogeneous layer, i.e., at $x=0,a$, play an important role. In accordance with Appendix A these values are given by

$$\left.\frac{\partial P_n}{\partial x}\right|_{x=x_j}=\frac{(-1)^{j+1}}{V_{Aj}}\int_0^t\left(\frac{d^2P_n(x_j)}{d\tau^2}+V_{Aj}^2\kappa_n^2P_n(x_j)\right)\times J_0[V_{Aj}\kappa_n(t-\tau)]\,d\tau, \tag{14}$$

where J_0 is the Bessel function of the zeroth order, $j=1,2$, the subscripts ''1'' and ''2'' refer to regions I and II, $x_1=0$, $x_2=a$, $\kappa_n=(k^2+n^2l^2)^{1/2}$, and $P_n(x_j)=P_n(x=x_j)$.

As it has been already stated, we neglect resistivity in regions I and II. Then, using Eq. (13) and the identity

$$1-\frac{z}{L}=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(nlz), \tag{15}$$

we obtain from Eq. (8), rewritten in terms of U, the equation

$$\frac{\partial^2U_n}{\partial t^2}+V_A^2n^2l^2U_n=-\frac{1}{\rho}\frac{\partial^2P_n}{\partial t\partial x}-\frac{2}{\pi n}\frac{\partial^2 f}{\partial t^2}. \tag{16}$$

The solution to this equation satisfying the initial conditions $U_n=\partial U_n/\partial t=0$ at $t=0$ is

$$U_n=-\int_0^t\left(\frac{2}{\pi n}\frac{\partial^2 f}{\partial\tau^2}+\frac{1}{\rho}\frac{\partial^2P_n}{\partial\tau\partial x}\right)\frac{\sin[V_Anl(t-\tau)]}{V_Anl}\,d\tau. \tag{17}$$

This expression together with Eq. (14) determines U_n at $x=0,a$. Taking $x=x_j$ in Eq. (17), using Eq. (14), integrating by parts, and changing the order of integration in the second term on the right-hand side of the obtained expression for $U_n(x_j)$, we arrive at

$$U_n(x_j)=-\frac{2}{\pi n}\int_0^t\frac{\partial f}{\partial\tau}\cos[V_{Aj}nl(t-\tau)]\,d\tau+\frac{(-1)^j}{\rho_jV_{Aj}}\int_0^t\left(\frac{d^2P_n(x_j)}{d\tau^2}+V_{Aj}^2\kappa_n^2P_n(x_j)\right)d\tau\times\int_0^{t-\tau}J_0(V_{Aj}\kappa_n\theta)\cos[V_{Aj}nl(t-\tau-\theta)]\,d\theta. \tag{18}$$

In what follows we assume that the inhomogeneous layer is thin, $a\ll L$, and we consider motions with the characteristic time L/V_A. Then it follows from Eq. (11) that $\partial P/\partial x\sim P/L$. This estimate implies that $P(t,x,z)=A(t,z)+\mathcal{O}(a/L)$ in the inhomogeneous layer, and we can take P independent of x in the first-order approximation with respect to the small parameter a/L. This approximation significantly simplifies the analysis. It was first used by Hollweg[26] and subsequently by Hollweg and Yang[27] when studying resonant absorption of MHD surface waves in a thin inhomogeneous layer. Since P does not vary across the inhomogeneous layer, it is completely determined by its values at the layer boundaries, $A(t,z)=P(t,0,z)=P(t,a,z)$. Now Eq. (10) is an equation with the right-hand side known. It determines $\partial u/\partial x$ for $0<x<a$. Substituting U and A expanded as Fourier series with respect to z into Eq. (10), rewritten in terms of U, we arrive at

$$\left(\frac{\partial^2}{\partial t^2}+(V_Anl)^2\right)\frac{\partial^2U_n}{\partial t\partial x}+\eta(V_Anl)^2\frac{\partial^3U_n}{\partial x^3}=-\frac{1}{\rho V_A^2}\left(\frac{d^2}{dt^2}+V_A^2\kappa_n^2\right)\frac{d^2A_n}{dt^2}. \tag{19}$$

When deriving this equation we have taken into account that the variation of f across the inhomogeneous layer can be neglected. Let us solve Eq. (19). Since resistivity is assumed to be weak, we can try to neglect the last term on the left-hand side of Eq. (19). However, if we then calculate the solution to the obtained ideal equation and substitute it into the neglected term, we find that this term has unbounded growth with time and eventually becomes large. This occurs because the solution to the ideal equation describes oscillations of each magnetic field line with its own frequency $V_A(x)nl$. Since the oscillation frequencies of different magnetic field lines are different, oscillations of neighboring magnetic field lines get more and more out of phase. This process, called phase mixing, leads to creation of large gradients in the x direction. Hence, to obtain the uniformly valid solution for large periods of time, we have to take the dissipative term in Eq. (19) into account.

To obtain the uniformly valid solution we use the Wentzel–Kramers–Brillouin (WKB) method. The crucial step is to find the uniformly valid approximate Green's function $G_n(t,\tau,x)$ with respect to time. And to find Green's function we have, in turn, to calculate the uniformly valid general solution to the homogeneous counterpart of Eq. (19). In accordance with the WKB method we introduce the ''stretched'' time $T=\epsilon t$, where the small parameter ϵ will be determined later. Then we look for the solution in the form

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$\partial U_n/\partial x = Q(T,x)\exp[i\epsilon^{-1}\Theta(T,x)]$. Substituting this ansatz into the homogeneous counterpart of Eq. (19), we arrive at

$$Q\frac{\partial\Theta}{\partial T}\left[\left(\frac{\partial\Theta}{\partial T}\right)^2-(V_{\rm A}nl)^2\right]-3i\epsilon\frac{\partial Q}{\partial T}\left(\frac{\partial\Theta}{\partial T}\right)^2-3i\epsilon Q\frac{\partial\Theta}{\partial T}\frac{\partial^2\Theta}{\partial T^2}+i\epsilon(V_{\rm A}nl)^2\frac{\partial Q}{\partial T}-i\epsilon^{-2}\eta(V_{\rm A}nl)^2Q\left(\frac{\partial\Theta}{\partial x}\right)^2=\mathcal{O}(\epsilon^2)+\mathcal{O}(\epsilon^{-1}{\rm R}^{-1}), \tag{20}$$

where the magnetic Reynolds number is given by $\mathrm{R}=aV_{\rm A1}/\eta$ with $V_{\rm A1}=V_{\rm A}(0)$. Now we impose the condition that the dissipative term on the left-hand side of this equation, which is the last term, be the same order as the terms proportional to ϵ. This condition results in $\epsilon=\mathrm{R}^{-1/3}$. Collecting the terms of the order unity in Eq. (20), we obtain the equation corresponding to the approximation of geometrical optics (e.g., Ref. 28). The solutions to this equation are

$$\Theta=\pm V_{\rm A}nlT, \quad \Theta=0. \tag{21}$$

Collecting the terms of the order ϵ in Eq. (20), we obtain the equation corresponding to the approximation of physical optics. Choosing either the first or the second expression Eq. (21) for Θ, we write this equation in the form

$$\frac{\partial Q}{\partial T}=-3\Lambda n^2T^2Q \quad \text{or} \quad \frac{\partial Q}{\partial T}=0, \tag{22}$$

where $\Lambda=(1/6)al^2V_{\rm A1}(dV_{\rm A}/dx)^2$. The solutions to these equations are straightforward, and eventually we obtain the general solution to the homogeneous counterpart of Eq. (19) in the form

$$\frac{\partial U_n}{\partial x}=\exp(-\Lambda n^2t^3/\mathrm{R})[C_1(x)\cos(V_{\rm A}nlt)+C_2(x)\sin(V_{\rm A}nlt)]+C_3(x), \tag{23}$$

where $C_1(x)$, $C_2(x)$, and $C_3(x)$ are arbitrary functions. Green's function $\mathrm{G}_n(t,\tau,x)$ with respect to time has to satisfy the following conditions: it is a solution to the homogeneous counterpart of Eq. (19) for any fixed $\tau<t$; $\mathrm{G}_n(t,\tau,x)=0$ for $t<\tau$; $\mathrm{G}_n(t,\tau,x)$ and $\partial\mathrm{G}_n/\partial t$ are continuous at $t=\tau$; $\partial^2\mathrm{G}_n/\partial t^2\to 1$ as $t\to\tau+0$. It is straightforward to check that the function $\mathrm{G}_n(t-\tau,x)$ determined by

$$\mathrm{G}_n(t,x)=(V_{\rm A}nl)^{-2}\mathrm{H}(t)[1-\exp(-\Lambda n^2t^3/\mathrm{R})\cos(V_{\rm A}nlt)]+\mathcal{O}(\mathrm{R}^{-1/3}) \tag{24}$$

satisfies all these conditions. Here $\mathrm{H}(t)$ is the Heaviside step function, $\mathrm{H}(t)=1$ for $t>0$ and $\mathrm{H}(t)=0$ for $t<0$. Note that G_n depends on the difference $t-\tau$ instead of t and τ separately because the coefficients of Eq. (19) are independent of time and, consequently, it is invariant with respect to the time shift. The solution to Eq. (19), satisfying the zero initial conditions, is given by the convolution of $\mathrm{G}_n(t,x)$ and the right-hand side of Eq. (19). Neglecting the second term in Eq. (24), which is small due to assumption $\mathrm{R}\gg 1$, and using integration by parts, we write this solution in the form

$$\frac{\partial U_n}{\partial x}=-\frac{1}{\rho V_{\rm A}^2}\int_0^t\exp[-\Lambda n^2(t-\tau)^3/\mathrm{R}]\times\cos[V_{\rm A}nl(t-\tau)]Y_n(\tau,x)\,d\tau, \tag{25}$$

where

$$Y_n(t,x)=\frac{d^2A_n}{dt^2}+V_{\rm A}^2\kappa_n^2A_n. \tag{26}$$

Integrating Eq. (25) with respect to x from 0 to a we calculate $U_n(a)-U_n(0)$. On the other hand, we can calculate this quantity using Eq. (18). Comparing the two expressions we obtain the governing equation for $A_n(t)$:

$$\sum_{j=1}^{2}V_{\rm Aj}\int_0^tY_n(\tau,x_j)\,d\tau\int_0^{t-\tau}J_0(V_{\rm Aj}\kappa_n\theta)\times\cos[V_{\rm Aj}nl(t-\tau-\theta)]\,d\theta+\int_0^a dx\int_0^tY_n(\tau,x)\times\exp[-\Lambda n^2(t-\tau)^3/\mathrm{R}]\cos[V_{\rm A}nl(t-\tau)]\,d\tau=\frac{2\rho V_{\rm A}^2}{\pi n}\sum_{j=1}^{2}(-1)^j\int_0^t\frac{\partial f}{\partial\tau}\cos[V_{\rm Aj}nl(t-\tau)]\,d\tau. \tag{27}$$

To solve this equation we use the Laplace transform. Using the theorem about the Laplace transform of convolution and a standard table of Laplace transforms (e.g., Ref. 29) we obtain the following solution to Eq. (27):

$$A_n(t)=\frac{i\rho V_{\rm A}^2}{\pi^2n}\int_{-\infty+\varsigma}^{\infty+\varsigma}\frac{W_n(\omega)}{D_n(\omega;\mathrm{R})}e^{-i\omega t}\,d\omega, \tag{28}$$

$$W_n(\omega)=\omega\hat{f}(\omega)\left(\frac{1}{(V_{\rm A1}nl)^2-\omega^2}-\frac{1}{(V_{\rm A2}nl)^2-\omega^2}\right), \tag{29}$$

where $\hat{f}(\omega)$ is the Laplace transform of $f(t)$, and ς is an arbitrary positive constant. The dispersion function $D_n(\omega;\mathrm{R})$ is given by

$$D_n(\omega;\mathrm{R})=D_n^{(0)}(\omega)+D_n^{(1)}(\omega;\mathrm{R}), \tag{30}$$

$$D_n^{(0)}(\omega)=\frac{V_{\rm A1}(V_{\rm A1}^2\kappa_n^2-\omega^2)^{1/2}}{(V_{\rm A1}nl)^2-\omega^2}+\frac{V_{\rm A2}(V_{\rm A2}^2\kappa_n^2-\omega^2)^{1/2}}{(V_{\rm A2}nl)^2-\omega^2}, \tag{31}$$

$$D_n^{(1)}(\omega;\mathrm{R})=i\int_0^a\frac{V_{\rm A}^2\kappa_n^2-\omega^2}{2\omega\delta_\omega}\{\mathbf{F}[(\omega-V_{\rm A}nl)/\delta_\omega]+\mathbf{F}[(\omega+V_{\rm A}nl)/\delta_\omega]\}\,dx, \tag{32}$$

where $\delta_\omega=(3\Lambda n^2/\mathrm{R})^{1/3}$. The incomplete F function is determined by

$$F(y,t)=\int_0^t\exp\left(iys-\frac{1}{3}s^3\right)ds, \tag{33}$$

and the complete F function is given by $\mathbf{F}(y)=F(y,\infty)$. This function was first introduced by Boris,[30] and then used by Goossens *et al.*[6] to describe the wave motions in stationary dissipative layers.

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The expression (28) determines the time evolution of the magnetic pressure in the inhomogeneous layer. It will be used in what follows to study dissipative eigenmodes and transition to the steady state of driven oscillations.

IV. GLOBAL MODES

Global modes of thin inhomogeneous layers, which are dissipative eigenmodes with the imaginary part of the eigenfrequency much smaller than the real part, have been studied analytically and numerically by many authors. In all previous studies it was assumed from the very beginning that perturbations of all variables were proportional to $\exp(-i\omega t)$. This resulted in an eigenvalue problem with ω^2 as an eigenvalue. This approach was used by, e.g., Einaudi and Mok,[31,32] Mok and Einaudi,[33] and Ruderman *et al.*[34,35] In this article we use another approach and calculate the eigenfrequency of the global mode using the nonstationary solution (28).

The dispersion equation determining frequencies of dissipative eigenmodes is $D_n(\omega;\mathrm{R})=0$. It is straightforward to see that $D_n^{(1)}(\omega;\mathrm{R})\sim(a/L)D_n^{(0)}(\omega)$. Since $a/L\ll 1$, this observation enables us to use the regular perturbation method and look for the solution to the dispersion equation in the form $\omega=\omega_n^{(0)}+\omega_n^{(1)}$ with $\omega_n^{(1)}/\omega_n^{(0)}\sim a/L$. In the first-order approximation we obtain $D_n^{(0)}(\omega_n^{(0)})=0$. This equation coincides with the dispersion equation for surface waves on a true magnetic interface in a cold ideal plasma. The solution to this equation is[36,37]

$$(\omega_n^{(0)})^2=\tfrac{1}{2}\{(V_{A1}^2+V_{A2}^2)\kappa_n^2-[(V_{A1}^2-V_{A2}^2)^2\kappa_n^4+4(V_{A1}V_{A2}\kappa^2)^2]^{1/2}\}. \tag{34}$$

In what follows we take $\omega_n^{(0)}>0$. In the second-order approximation we obtain

$$\omega_n^{(1)}=-D_n^{(1)}(\omega_n^{(0)};\mathrm{R})\left(\frac{dD_n^{(0)}}{d\omega}\right)^{-1}, \tag{35}$$

where the derivative on the right-hand side of this expression is calculated at $\omega=\omega_n^{(0)}$. Let us calculate the asymptotic expression for $D_n^{(1)}(\omega_n^{(0)};\mathrm{R})$ for $\mathrm{R}\gg 1$. For brevity we write ω instead of $\omega_n^{(0)}$ when doing this calculation. Using integration by parts, we obtain

$$\mathbf{F}[(\omega\pm V_A nl)/\delta_\omega]=\frac{i\delta_\omega}{\omega\pm V_A nl}+\mathcal{O}(\mathrm{R}^{-4/3}). \tag{36}$$

While this formula with the plus sign is valid for all values of x (recall that $\omega>0$ and δ_ω is of the order $\mathrm{R}^{-1/3}$), this formula with the minus sign is valid only when the denominator of the first term on the right-hand side is not small, i.e., when $|\omega-V_A nl|\gg\delta_\omega$. This condition is violated in the vicinity of the Alfvén resonant position x_A determined by the condition $V_A(x_A)nl=\omega$. In the vicinity of this position we can use the approximate expression

$$\omega-V_A nl\approx\frac{\Delta(x-x_A)}{2\omega},\quad \Delta=-(nl)^2\left.\frac{dV_A^2}{dx}\right|_{x=x_A}. \tag{37}$$

The condition $|\omega-V_A nl|\sim\delta_\omega$ determines the dissipative layer embracing the ideal resonant position x_A (for more detailed discussion see, e.g., Ref. 6). Using this condition and Eq. (37), we obtain the thickness of the dissipative layer $\delta_A=|\omega\eta/\Delta|^{1/3}=2\omega\delta_\omega(x_A)/|\Delta|$. Now we introduce the quantity s_A such that $\delta_A\ll s_A\ll a$. Then we use Eq. (36) for $|x-x_A|>s_A$, and Eq. (37) for $|x-x_A|<s_A$, to rewrite Eq. (32) in the form

$$\begin{aligned}D_n^{(1)}(\omega;R)={}&-\frac{1}{2\omega}\int_0^a\frac{V_A^2\kappa_n^2-\omega^2}{\omega+V_A nl}dx\\&-\frac{1}{2\omega}\left(\int_0^{x_A-s_A}+\int_{x_A+s_A}^a\right)\\&\times\frac{V_A^2\kappa_n^2-\omega^2}{\omega-V_A nl}dx+\frac{i\omega^2k^2}{\delta_A|\Delta|(nl)^2}\\&\times\int_{x_A-s_A}^{x_A+s_A}\mathbf{F}[(x-x_A)\,\mathrm{sign}\,(\Delta)/\delta_A]\,dx.\end{aligned} \tag{38}$$

It is convenient to make the variable substitution $x'=(x-x_A)\,\mathrm{sign}\,(\Delta)/\delta_A$ in the last integral. Now we consider $R\to\infty$. This enables us to take $s_A\to+0$ and $\delta_A\to+0$. Then the second integral tends to the principal Cauchy part of the integral over $[0,a]$. The third integral tends to $\int_{-\infty}^{\infty}\mathbf{F}(x)\,dx=\pi$. As a result we arrive at

$$D_n^{(1)}(\omega;\mathrm{R})=\mathcal{P}\int_0^a\frac{V_A^2\kappa_n^2-\omega^2}{V_A^2n^2l^2-\omega^2}dx+\frac{\pi i\omega^2k^2}{|\Delta|(nl)^2}, \tag{39}$$

where $\mathcal{P}$ indicates the principal Cauchy part of an integral. This result enables us to calculate the explicit expression for $\omega_n^{(1)}$ in terms of equilibrium quantities. The real part of this quantity provides only small correction to the real part of the eigenfrequency, and it is not important for what follows. In contrast, the imaginary part is of great importance because it describes wave damping. Using Eqs. (31), (35), and (39), we obtain for the wave decrement $\gamma_n=-\Im(\omega_n^{(1)})$ ($\Im$ indicates the imaginary part of a quantity) the following expression

$$\gamma_n=\frac{\pi\omega k^2}{|\Delta|(nl)^2}\left(\sum_{j=1}^{2}\frac{V_{Aj}[V_{Aj}^2(k^2+\kappa_n^2)-\omega^2]}{(V_{Aj}^2n^2l^2-\omega^2)^2(V_{Aj}^2\kappa_n^2-\omega^2)^{1/2}}\right)^{-1}. \tag{40}$$

In this expression $\omega=\omega_n^{(0)}$. The important property is that $\gamma_n=\mathcal{O}(a/L)$, and it is independent of R.

V. TRANSITION TO THE STEADY STATE OF GLOBAL DRIVEN OSCILLATION

We use the term ''global motion'' or ''global oscillation'' for a coherent motion of the cavity ''as a whole'' in the x direction. A global mode is a particular case of global oscillation. In what follows we assume that the harmonic driving starts at the initial moment of time. To satisfy the condition that $f=\partial f/\partial t=0$ at $t=0$ we take

$$f=f_0(1-e^{-\Omega t})\sin(\Omega t). \tag{41}$$

The amplitude f_0 is constant for $0<x<a$, and we do not specify its dependence on x outside of this interval because it is not important for what follows. We assume that Ω is in the Alfvén continuum of the fundamental harmonic with respect to z, $V_{A1}l<\Omega<V_{A2}l$. We also assume that $\Omega<V_{A1}\kappa_1$, so

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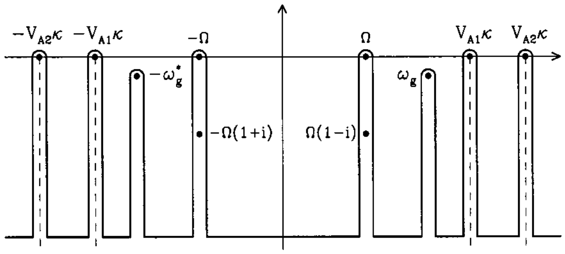

FIG. 2. The integration contour in the complex ω-plane. The boldface points show the simple poles and the branch points. The dashed lines show the cuts.

that the motion in regions I and II is evanescent in the x direction. The analysis is the same for each harmonic with respect to z. For brevity we drop the subscript n, indicating the number of a harmonic. In particular, we write κ and A instead of κ_n and A_n.

Our aim is to investigate the asymptotic behavior of the perturbation of the magnetic pressure in the inhomogeneous layer, $A(t)$, for $\Omega t \gg 1$. The temporal evolution of $A(t)$ is determined by Eq. (28). The function $D^{(0)}(\omega)$, given by Eq. (31), has four branch points, which are $\omega = \pm V_{A1}\kappa$ and $\omega = \pm V_{A2}\kappa$. These branch points are shown by the boldface points in Fig. 2. To obtain the single-valued branch of this function, we make cuts from these branch points to $-i\infty$. These cuts are shown in Fig. 2 by dashed lines. In the complex plane with these cuts $D^{(0)}(\omega)$ is a single-valued function, and so is the function W/D in Eq. (28).

The Laplace transform of the function $f(t)$ is

$$\hat{f} = f_0\Omega\left(\frac{1}{\Omega^2-\omega^2} - \frac{1}{\Omega^2-(\omega+i\Omega)^2}\right). \tag{42}$$

Then we conclude that W/D has simple poles at $\omega = \pm\Omega$ and $\omega = \Omega(\pm 1 - i)$. In accordance with the results of the previous section, there is also a pole at the frequency of the global mode $\omega_g \approx \omega^{(0)} + \omega^{(1)}$. It is straightforward to show that $D(-\omega^*) = D^*(\omega)$, where the asterisk indicates the complex conjugate value. This relation implies that, if $D(\omega_g) = 0$, then $D(-\omega_g^*) = 0$, and there is an additional pole of W/D at $-\omega_g^*$. It seems from the first sight that, since $W(\omega)$ has poles at $\omega = \pm V_{A1}nl$ and $\omega = \pm V_{A2}nl$, there are poles of W/D at $\omega = \pm V_{A1}nl$ and $\omega = \pm V_{A2}nl$. However, more careful examination shows that the function $D(\omega)$ also has poles at this values of ω. The ratio of the two functions, W and D, each having simple poles at $\omega = \pm V_{A1}nl$ and $\omega = \pm V_{A2}nl$, is regular at $\omega = \pm V_{A1}nl$ and $\omega = \pm V_{A2}nl$. The six simple poles of the integrand are shown by the boldface points in Fig. 2.

To calculated $A(t)$ we deform the integration contour as shown in Fig. 2. It is assumed that the horizontal parts of the contour have been moved to $-i\infty$, so they do not contribute to the integral. The contributions from the parts of contour going along the cuts describe the transition to the steady state of oscillation in regions I and II. It can be shown with the use of integration by parts that, for $\Omega t \gg 1$, these contributions are of the order $(\Omega t)^{-1}$, so they can be neglected in our asymptotic analysis. The contributions from the poles $\Omega(\pm 1 - i)$ are proportional to $e^{-\Omega t}$ and also can be neglected. It is possible that, in addition to ω_g and $-\omega_g^*$, there are other zeros of the dispersion function $D(\omega;\mathrm{R})$. However, they have imaginary parts of the same order as real parts, so that, similar to the poles at $\Omega(\pm 1 - i)$, their contributions can be neglected. Hence, the significant contributions come only from the four poles, $\pm\Omega$, ω_g, and $-\omega_g^*$, and asymptotically the integral in Eq. (28) is equal to the sum of residues with respect to these poles multiplied by $-2\pi i$. These residues are equal to $\tilde{A}_\Omega e^{-i\Omega t}$, $\tilde{A}_\Omega^* e^{i\Omega t}$, $\tilde{A}_g e^{-i\omega_g t}$, and $\tilde{A}_g^* e^{i\omega_g^* t}$, respectively, where

$$\tilde{A}_\Omega = \frac{-f_0\Omega n^2 l^2 (V_{A2}^2 - V_{A1}^2)}{2D(\Omega;\mathrm{R})(V_{A1}^2 n^2 l^2 - \Omega^2)(V_{A2}^2 n^2 l^2 - \Omega^2)}, \tag{43}$$

$$\tilde{A}_g = \frac{f_0\Omega n^2 l^2 \omega_g (V_{A2}^2 - V_{A1}^2)}{(V_{A1}^2 n^2 l^2 - \omega_g^2)(V_{A2}^2 n^2 l^2 - \omega_g^2)} \times\left(\frac{1}{\Omega^2-\omega_g^2} - \frac{1}{\Omega^2-(\omega_g+i\Omega)^2}\right)\left(\frac{dD}{d\omega}\right)^{-1}. \tag{44}$$

The quantity $dD/d\omega$ in Eq. (44) is calculated at $\omega = \omega_g$. The asymptotic behavior of $A(t)$ is described by

$$A(t) = \Re(A_\Omega e^{-i\Omega t}) + e^{-\gamma t}\Re(A_g e^{-i\omega_r t}), \tag{45}$$

$$A_{\Omega,g} = 4(\pi n)^{-1}\rho V_A^2 \tilde{A}_{\Omega,g}, \tag{46}$$

where $\omega_r = \Re(\omega_g) \approx \omega^{(0)}$, and $\Re$ indicates the real part of a quantity. We see that, for $\Omega t \gg 1$, the global oscillation of the cavity is described by the same equation as the oscillation of a weakly damped oscillator with the frequency ω_r and the decrement γ, which was first suggested by Hollweg.[26] In particular, the characteristic time of transition to the steady state of oscillation is of the order γ^{-1}. The important property is that this time is independent of R for $\mathrm{R} \gg 1$.

When $|\Omega - \omega_r| \gg \gamma$, the dimensionless amplitude of the magnetic pressure oscillation in the steady state, $A_\Omega/\rho V_A^2$, is of the order of the dimensionless amplitude of the velocity oscillation at the boundary, $f_0 l/\Omega$. When $|\Omega - \omega_r| \lesssim \gamma$, the driver is in quasi-resonance with the cavity. We can use the approximate formula

$$D(\Omega;\mathrm{R}) \approx (\Omega - \omega_g)\left.\frac{dD}{d\omega}\right|_{\omega=\omega_g}, \tag{47}$$

and $A_\Omega \approx -A_g$. With the aid of Eq. (47), we obtain from Eqs. (43) and (46) that

$$A_\Omega/\rho V_A^2 \sim f_0 l/|\Omega - \omega_g| \sim (f_0 l/\Omega)(L/a) \gg f_0 l/\Omega.$$

VI. LOCAL MOTIONS

Let us study the behavior of the x and y components of the velocity, u and v. First we study the nonresonant driving where $|\Omega - \omega_r| \gg \gamma$. It is interesting that this definition of nonresonant driving coincides with that put forward on qualitative grounds by Allan *et al.*[38] To obtain the expression for v_n, we note that Eq. (9), determining v, and Eq. (10), determining $\partial u/\partial x$, differ only in their right-hand sides. Then it immediately follows that the left-hand side of the equation for v_n coincides with the left-hand side of Eq. (19) for $\partial U_n/\partial x$, while its right-hand side is $-(ik/\rho)d^2A_n/dt^2$.

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Therefore, it is clear that the expression for v_n is obtained from Eq. (25) for $\partial U_n/\partial x$ by substituting $(ik/\rho)A_n$ for $(1/\rho V_A^2)Y_n$. As a result, after the substitution $\tau \to t-\tau$ in the integral, we arrive at

$$v_n = -\frac{ik}{\rho}\int_0^t A_n(t-\tau)\exp(-\Lambda n^2\tau^3/\mathrm{R})\cos(V_A nl\tau)\,d\tau. \tag{48}$$

Once again, we drop the subscript ''n'' for brevity.

To study the asymptotic behavior of v for $\Omega t \gg 1$, we substitute Eq. (28) into Eq. (48), change the order of integration, and make the substitution of the integration variable $\tau = s/\delta_\omega$ [recall that $\delta_\omega = (3\Lambda n^2/\mathrm{R})^{1/3}$]. As a result we obtain

$$v = \frac{kV_A^2}{\pi^2 n\delta_\omega}\int_{-\infty+s}^{\infty+s}\frac{W_n(\omega)}{D_n(\omega;\mathrm{R})}e^{-i\omega t}\,d\omega \times \int_0^{t\delta_\omega}\exp\left(i\omega s/\delta_\omega - \frac{1}{3}s^3\right)\cos(V_A nls/\delta_\omega)\,ds. \tag{49}$$

Then we use the same procedure as when obtaining Eq. (45). However, now we cannot neglect the contributions from the poles at $\omega = \Omega(\pm 1 - i)$, because they do not decay exponentially on the time scale Ω^{-1}. Using integration by parts to simplify the contribution from these poles, and neglecting small terms of the order of δ_ω/Ω, we eventually arrive at the asymptotic expression valid for $t\Omega \gg 1$

$$v = -\frac{ik}{\rho}\Re\left\{\delta_\omega^{-1}A_\Omega e^{-i\Omega t}\int_0^{t\delta_\omega}\exp\left(i\Omega_s/\delta_\omega - \frac{1}{3}s^3\right)\times\cos(V_A nls/\delta_\omega)\,ds + \delta_\omega^{-1}A_g e^{-\tau - i\omega_r t}\times\int_0^{t\delta_\omega}\exp\left[(\gamma + i\omega_r)s/\delta_\omega - \frac{1}{3}s^3\right]\cos(V_A nls/\delta_\omega)\,ds + A_{in}e^{-(t\delta_\omega)^3}\cos(V_A nlt)\right\}. \tag{50}$$

The third term in the curly brackets appears due to the contributions from the poles at $\omega = \Omega(\pm 1 - i)$. We do not give the expression for A_{in} because it is not used in what follows, and only note that A_{in} is of the same order of magnitude as A_Ω and A_g.

As it has been pointed out in Sec. IV, the dispersion function $D_n(\omega;\mathrm{R})$ might have additional zeros besides ω_g and $-\omega_g^*$. All these additional zeros have the property that their imaginary parts are negative and of the order of real parts. Contributions from these hypothetical additional zeros would give additional terms in Eq. (50) similar to the third term in the curly brackets. However, neither this third term nor the hypothetical additional terms are of particular importance. They simply describe oscillations with the local Alfvén frequency decaying on the time scale $\Omega^{-1}\mathrm{R}^{1/3}$. Hence the possible presence of these additional terms does not affect the following analysis.

To obtain the steady state of oscillations we take $t \to \infty$. Then the second and third terms in the curly brackets exponentially decrease, and we arrive at

$$v = -\frac{ik}{\rho\delta_\omega}\Re\left\{A_\Omega e^{-i\Omega t}\int_0^\infty\exp\left(i\Omega s/\delta_\omega - \frac{1}{3}s^3\right)\times\cos(V_A nls/\delta_\omega)\,ds\right\}. \tag{51}$$

Using integration by parts, it is straightforward to show that $v \sim f_0$ for any overtone $(n>1)$. For the fundamental harmonic $(n=1)$ this estimate is valid everywhere except the narrow dissipative layer with the thickness δ_A (see Sec. IV), embracing the ideal resonant position x_A determined by $lV_A(x_A) = \Omega$. In this dissipative layer we use the approximation given by Eq. (37), and arrive at the well-known asymptotic formula for the fundamental harmonic (e.g., Ref. 6)

$$v = -\frac{ik}{2\rho\delta_\omega}\Re\{A_\Omega e^{-i\Omega t}\mathbf{F}(\sigma\,\mathrm{sign}(\Delta))\}, \tag{52}$$

with $\sigma = (x - x_A)/\delta_A$ and $\delta_A = |\eta\Omega/\Delta|^{1/3}$. Since $\mathbf{F}(y) \sim 1$ for $|y| \sim 1$, we conclude that the amplitude of v in the dissipative layer is of the order $f_0\Omega/\delta_\omega$. Assuming that $V_{A1} \sim V_{A2}$ and $dV_A/dx \sim V_{A1}/a$, we obtain the estimate $\delta_\omega \sim \Omega(al\mathrm{R})^{-1/3}$. Hence, once again, we arrive at the well-known result that the amplitude of v in the dissipative layer is scaled as $\mathrm{R}^{1/3}$.

Let us now estimate the transitional time to the steady state of oscillations t_{tr}. Using integration by parts, we obtain from Eq. (50)

$$v = -\frac{ik}{\rho}\Re\left\{\frac{iA_\Omega\Omega}{W_\Omega}e^{-i\Omega t} + \frac{A_g(\gamma + i\omega_r)}{W_g}e^{-\gamma t - i\omega_r t} - \left[\left(\frac{iA_\Omega\Omega}{W_\Omega} + \frac{A_g(\gamma + i\omega_r)}{W_g} - A_{in}\right)\cos(V_A lt) + \left(\frac{A_\Omega}{W_\Omega} + \frac{A_g}{W_g}\right)\sin(V_A lt)\right]e^{-(t\delta_\omega)^3/3}\right\} + \mathcal{O}(\delta_\omega/\Omega), \tag{53}$$

where

$$W_\Omega = \Omega^2 - V_A^2 n^2 l^2, \quad W_g = (\omega_r - i\gamma)^2 - V_A^2 n^2 l^2. \tag{54}$$

The characteristic damping times of the second and third terms in the curly brackets are γ^{-1} and $\delta_\omega^{-1} \sim \Omega^{-1}\mathrm{R}^{1/3}$, respectively. Then t_{tr} is the largest of the two times,

$$t_{\mathrm{tr}} = \max(\gamma^{-1}, \Omega^{-1}\mathrm{R}^{1/3}). \tag{55}$$

The asymptotic expression (53) is not valid when x is close to either x_g, determined by $nlV_A(x_g) = \omega_r$, or to x_A for the fundamental harmonic. When x is close to x_A for the fundamental harmonic, the contribution to Eq. (53) from the second integral in Eq. (50) (terms proportional to A_g) remains the same. To evaluate the first integral we use the approximate formula (37) with $\omega = \Omega$. As a result, we obtain the following asymptotic expression for the first term in the curly brackets in Eq. (50):

$$\tfrac{1}{2}\delta_\omega^{-1}A_\Omega e^{-i\Omega t}F(\sigma\,\mathrm{sign}(\Delta), t\delta_\omega). \tag{56}$$

First of all, we note that $F(\sigma, t\delta_\omega) \approx t\delta_\omega + \frac{1}{2}i\sigma(t\delta_\omega)^2$ for $|\sigma|t\delta_\omega \ll 1$ (for simplicity we take $\Delta > 0$). Hence, the amplitude of oscillations at the resonant position $(\sigma = 0)$ grows secularly with time at the initial stage when dissipation is

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negligible. The characteristic spatial scale created by phase mixing at the time t is determined by the condition that the second term in this expression is of the order of the first term. This condition gives $|\sigma|\sim(t\delta_\omega)^{-1}$, or $|x-x_A|\sim|tldV_A/dx|^{-1}$. In the magnetospheric context these results were first obtained by Wright,[39,40] and then confirmed by Mann *et al.*[41]

The function $F(y,t)$ converges to $\mathbf{F}(y)$ very rapidly as $t\to\infty$. For example, $F(0,2)$ differs from $\mathbf{F}(0)$ by less that 2%. If we take $t\sim\Omega^{-1}R^{1/3}$, then $t\delta_\omega\sim(al)^{-1/3}$, and $F(\sigma\,\mathrm{sign}(\Delta),t\delta_\omega)\approx\mathbf{F}(\sigma\,\mathrm{sign}(\Delta))$ for $a/L\lesssim0.1$. Hence, we conclude that the characteristic transitional time for the first term in the curly brackets in Eq. (50) is $\Omega^{-1}R^{1/3}$, and Eq. (55) is also valid for x close to x_A. Using the estimate for the characteristic spatial scale created by phase mixing at the time $t,|tldV_A/dx|^{-1}$, and the estimate $|dV_A/dx|\sim V_A/a$, we obtain that this transitional time is just the time necessary for phase mixing to create the spatial scale of the order of $aR^{-1/3}$.

When x is close to x_g, the contribution into Eq. (53) from the first integral in Eq. (50) (terms proportional to A_Ω) remains the same. To evaluate the second integral we once again use Eq. (37), but now with $\omega=\omega_r$, and with x_g substituted for x_A. As a result, we obtain the following asymptotic expression for the second term in the curly brackets in Eq. (50):

$$\tfrac{1}{2}\,\delta_\omega^{-1}A_g e^{-\gamma t-i\omega_r t}F(\sigma\,\mathrm{sign}(\Delta)-i\gamma/\delta_\omega,t\delta_\omega). \tag{57}$$

Note that here $\sigma=(x-x_g)/\delta_A$, and δ_A and δ_ω are calculated at $x=x_g$. Let us first consider the case where $\gamma\lesssim\Omega R^{-1/3}$. Then $\gamma/\delta_\omega\lesssim1$ and, similar to Eq. (56), we can substitute $\mathbf{F}(\sigma\,\mathrm{sign}(\Delta)-i\gamma/\delta_\omega)$ for $F(\sigma\,\mathrm{sign}(\Delta)-i\gamma/\delta_\omega,t\delta_\omega)$ when $t\Omega\gtrsim R^{1/3}$. We see that the contribution from the second integral in Eq. (50) is of the order Ω/δ_ω. Since, in accordance with Eq. (53), the contribution from the first integral is of the order unity, it can be neglected. As a result, we obtain that, for $t\Omega\gtrsim R^{1/3}$, the asymptotic behavior of v in the dissipative layer embracing x_g is given by

$$v=-\frac{ik}{2\rho\delta_\omega}\Re\{A_g e^{-\gamma t-i\omega_r t}\mathbf{F}(\sigma\,\mathrm{sign}(\Delta)-i\gamma/\delta_\omega)\}. \tag{58}$$

The behavior of the function $\mathbf{F}(y-i\alpha)$ with $\alpha>0$ was studied by Ruderman *et al.*[34] and Tirry and Goossens.[22] In particular, it was shown that, for $\alpha\lesssim1$, this behavior is almost the same as for $\alpha=0$. Hence, we conclude that, after the transitional time of the order $\Omega^{-1}R^{1/3}$, the quasi-stationary dissipative layer embracing x_g is formed. The structure of this dissipative layer is practically the same as of that embracing x_A. The amplitude of oscillations is of the same order of magnitude as in the dissipative layer embracing x_A. Then, when the time progresses, the oscillations in the dissipative layer slowly decay on the time scale γ^{-1}. This decay is adiabatic in the sense that the structure of the dissipative layer remains the same. Note that the appearance of large-amplitude oscillations in the vicinity of x_g, even when $\Omega\neq\omega_r$, was found in numerical simulations (e.g., Refs. 24 and 25).

Let us now study the case where $\gamma\gg\Omega R^{-1/3}\sim\delta_\omega$. The important questions in studying the temporary evolution of the expression (57) are when it attains its maximum amplitude, and what this amplitude is. It is shown in Appendix B that, at $\sigma=0$, the maximum amplitude is attained at $t=t_m\approx3\gamma^{-1}\ln(\gamma/\delta_\omega)$. Using Eq. (B4), we conclude that the maximum amplitude of oscillations with the frequency ω_r at $\sigma=0$ is of the order $f_0\Omega/\gamma$. Hence, while it is still much larger than the driving amplitude f_0, it is much smaller than the amplitude of oscillations with the frequency Ω at $x=x_A$ in the steady state.

Let us take t satisfying

$$1<t\delta_\omega<(\gamma/\delta_\omega)^{1/2}\sim(al)^{2/3}R^{1/6}. \tag{59}$$

Then we can use Eq. (B6). Using this equation, we obtain for the amplitude of oscillations with the frequency ω_r at $\sigma=0$ the estimate $f_0(al)^{-1}e^{-(t\delta_\omega)^3/3}$. For $t=t_1=\delta_\omega^{-1}|3\ln(al)|^{1/3}$ this quantity equals f_0. For $al=0.1$ we obtain $t_1\delta_\omega\approx2$, so the condition (59) is satisfied for $\gamma/\delta_\omega\geqslant4$. Since for $t>t_m$ the amplitude of oscillations monotonically decreases, it is even smaller than f_0 for $t>t_1$. And it is also smaller at $\sigma\neq0$ than at $\sigma=0$. Hence, we conclude that the characteristic damping time of oscillations with the frequency ω_r near x_g is t_1. Since $t_1\sim\delta_\omega^{-1}\sim\Omega R^{1/3}$, the expression (55) is once again valid.

The absolute value of the integrand in the expression for $F(\sigma\,\mathrm{sign}(\Delta)-i\gamma/\delta_\omega,t\delta_\omega)$ takes its maximum value at $s=(\gamma/\delta_\omega)^{1/2}$. Consequently, $F(\sigma\,\mathrm{sign}(\Delta)-i\gamma/\delta_\omega,t\delta_\omega)$ attains its limiting value of $\mathbf{F}(\sigma\,\mathrm{sign}(\Delta)-i\gamma/\delta_\omega)$ at $t\sim(\gamma/\delta_\omega^3)^{1/2}\sim al\Omega^{-1}R^{1/2}$. This result is in complete agreement with the behavior of the function $\mathbf{F}((x-x_g)/\delta_\omega-i\gamma/\delta_\omega)$ for $\gamma/\delta_\omega\gg1$. It has the form of a wave packet with the center at x_g, the characteristic width $a\gamma/\Omega$, and the period of the carrier wave of the order of $aR^{-1/2}$.[34,35] Scales of the order of $aR^{-1/2}$ are created by the phase mixing after the time of the order of $\Omega^{-1}R^{1/2}$. However, this structure cannot be observed in the vicinity of x_g. The reason is that $F(-i\gamma/\delta_\omega)\sim\exp(2/3(\gamma/\delta_\omega)^{3/2})$.[35] As a result, the amplitude of oscillations with frequency ω_r in the vicinity of x_g is proportional to $\exp(-\tfrac{1}{3}(\gamma/\delta_\omega)^{3/2})\sim\exp(-\tfrac{1}{3}R^{1/2}\gamma/\Omega)$ for $t\sim\Omega^{-1}R^{1/2}$. Since we consider the case where $\gamma/\Omega\gg R^{-1/3}$, this amplitude is exponentially small, so that oscillations with frequency ω_r are completely dominated by those with frequency Ω.

Note that the formation of the energy-containing layer, embracing x_g, with the characteristic width equal to $a\gamma/\Omega$, and with the amplitude of oscillations inside it of the order of $f_0\Omega/\gamma$, was predicted by Lee and Roberts[23] and Mann *et al.*[41] However, since these authors used the ideal MHD equations to describe plasma motions, they did not study damping of oscillations in the energy-containing layer, and did not find the minimum spatial scale at which dissipation stops phase mixing.

We start studying the behavior of U in the inhomogeneous layer from considering the motion of the layer boundaries, described by $U(t,x_j)$ (recall that $x_1=0$ and $x_2=a$). These quantities are determined by Eq. (18) with $P_n(x_j)$

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$=A_n(t)$. To obtain the expression describing the asymptotic behavior of $U(t,x_j)$, we make the Laplace transform of Eq. (18), which yields, with the aid of Eq. (28),

$$\hat{U}_j=\frac{2\omega^2\hat{f}}{\pi n(V_{\mathrm{A}j}^2n^2l^2-\omega^2)}+\frac{2(-1)^j\omega V_{\mathrm{A}j}W(\omega)(V_{\mathrm{A}j}^2\kappa_n^2-\omega^2)^{1/2}}{\pi nD(\omega;\mathrm{R})(V_{\mathrm{A}j}^2n^2l^2-\omega^2)}, \tag{60}$$

where $\hat{U}_j$ is the Laplace transform of $U(t,x_j)$, and $W(\omega)$ and $D(\omega;\mathrm{R})$ are given by Eqs. (29)–(32). It is straightforward to show that the right-hand side of Eq. (60) is regular at $\omega^2=V_{\mathrm{A}j}^2n^2l^2$. Now the asymptotic behavior of $U(t,x_j)$ for $t\Omega\gg1$ is calculated in the same way as that of $A(t)$. As a result we arrive at the expression, similar to Eq. (45),

$$U(t,x_j)=\Re(U_\Omega e^{-i\Omega t})+e^{-\gamma t}\Re(U_g e^{-i\omega_r t}), \tag{61}$$

where

$$U_\Omega=\frac{i\Omega A_\Omega\{V_{\mathrm{A}1}(V_{\mathrm{A}1}^2\kappa^2-\Omega^2)^{1/2}+V_{\mathrm{A}2}(V_{\mathrm{A}2}^2\kappa^2-\Omega^2)^{1/2}\}}{\rho V_{\mathrm{A}}^2n^2l^2(V_{\mathrm{A}1}^2-V_{\mathrm{A}2}^2)}, \tag{62}$$

$$U_g=\frac{i\omega_rA_g(V_{\mathrm{A}1}^2\kappa^2-\omega_r^2)^{1/2}}{\rho_1V_{\mathrm{A}1}(V_{\mathrm{A}1}^2n^2l^2-\omega_r^2)}. \tag{63}$$

When deriving this expression we have neglected small terms of the order of a/L. We drop the subscript j at U_Ω and U_g because these quantities take the same values for $j=1$ and $j=2$. We see that the transitional time to the steady state of oscillations of the inhomogeneous layer boundaries is the same as for $A(t)$, namely γ^{-1}.

The quantity $\partial U/\partial x$ is given by Eq. (25). To study the asymptotic behavior of $U(t,x)$ for $t\Omega\gg1$ we use the same procedure as in studying the asymptotic behavior of v. We substitute Eq. (45) into Eq. (25), integrate the result with respect to x, and use the boundary conditions at $x=x_j$. As a result we obtain

$$\begin{aligned}U(t,x)=U(t,x_j)+\int_{x_j}^{x}\frac{dx}{\rho V_{\mathrm{A}}^2\delta_\omega}\Re\bigg\{&A_\Omega(\Omega^2-V_{\mathrm{A}}^2\kappa^2)e^{-i\Omega t}\\&\times\int_0^{t\delta_\omega}\exp\left(i\Omega s/\delta_\omega-\frac{1}{3}s^3\right)\cos(V_{\mathrm{A}}ls/\delta_\omega)\,ds\\&+A_g(\omega_g^2-V_{\mathrm{A}}^2\kappa^2)e^{-\gamma t-i\omega_r t}\\&\times\int_0^{t\delta_\omega}\exp\left[(\gamma+i\omega_r)s/\delta_\omega-\frac{1}{3}s^3\right]\\&\times\cos(V_{\mathrm{A}}ls/\delta_\omega)\,ds\bigg\}.\end{aligned} \tag{64}$$

It can be verified that this formula gives the same result for $j=1$ and $j=2$. The investigation of this expression is similar to that of Eq. (50). It can be shown that the transitional time to the steady state of oscillations is t_{tr}. For the fundamental harmonic there is the dissipative layer embracing x_{A}. The steady state of oscillations in this layer is described by

$$U(t,x)=U(t,x_{\mathrm{A}})+\frac{k^2\omega}{\rho\Delta}\Re\{iA_\Omega e^{-i\Omega t}\mathbf{G}(\sigma\,\mathrm{sign}\,(\Delta))\}, \tag{65}$$

where the equilibrium quantities are calculated at $x=x_{\mathrm{A}}$. The incomplete G-function is determined by

$$\mathrm{G}(y,t)=\int_0^t\frac{\exp(iys)-1}{s}e^{-s^3/3}\,ds, \tag{66}$$

and the complete G-function, first introduced by Boris,[30] is given by $\mathbf{G}(y)=\mathrm{G}(y,\infty)$. Once again, Eq. (65) gives a well-known result (e.g., Ref. 6). The amplitude of $U(t,x)$ in the dissipative layer is of the order of $(af_0/L)\ln\mathrm{R}$ when $\ln\mathrm{R}>L/a$, and of the order of f_0 when $\ln\mathrm{R}\lesssim L/a$.

When $\gamma\ll\Omega\mathrm{R}^{-1/3}$, the quasi-stationary dissipative layer, similar to that embracing x_{A}, forms near x_g. On the other hand, when $\gamma\gg\Omega\mathrm{R}^{-1/3}$, oscillations with the frequency ω_r decay faster than this layer forms.

Let us now concentrate on the case where $\gamma\gg\Omega\mathrm{R}^{-1/3}$, and estimate the order of magnitude of different terms in Eq. (64) for x not very close to x_{A}, i.e., out of the dissipative layer. The analysis, similar to that used to calculate the maximum amplitude of v at x_g, leads to the conclusion that the maximum amplitude of the second term in the curly brackets at x_g is of the same order as at $x\neq x_g$. Let us first consider the fundamental harmonic. It is straightforward to show that the second term in Eq. (64) is of the order $alf_0\ll f_0$ for $x<x_{\mathrm{A}}$ when $j=1$, and for $x>x_{\mathrm{A}}$ when $j=2$. Since the term $U(t,x_j)$ is of the order f_0, we conclude that it is the dominant term in Eq. (64) for $x<x_{\mathrm{A}}$ when $j=1$, and for $x>x_{\mathrm{A}}$ when $j=2$. The situation is even simpler for the overtones. The term $U(t,x_j)$ is dominant in the whole interval $0<x<a$ both for $j=1$ and $j=2$. As we have seen, the transitional time to the steady state of oscillations with the frequency Ω is of the order γ^{-1} for $U(t,x_j)$. Consequently, we obtain a very important result. The transitional time to the steady state of oscillations for the dominant motion in the x direction far away from the dissipative layer is γ^{-1}. When $\gamma\gg\Omega\mathrm{R}^{-1/3}$, this transitional time is much smaller than t_{tr}. After a time of the order of γ^{-1}, the inhomogeneous layer oscillates ''as a whole'' in x direction with the frequency Ω, with the exception of the narrow dissipative layer embracing x_{A}. In this layer the transitional time is $t_{\mathrm{tr}}\gg\gamma^{-1}$.

In the case of the resonant driving, where $|\Omega-\omega_r|\lesssim\gamma$, the analysis of this section remains the same for the overtones. For the fundamental harmonic the only difference is that amplitudes of all motions are larger by the factor of the order of L/a than those in the case of the nonresonant driving, and in all regimes there is only one dissipative layer embracing the position $x_{\mathrm{A}}\approx x_g$. It is also instructive to recall that, in the case of resonant driving, $A_g\approx-A_\Omega$ for the fundamental harmonic.

VII. ENERGY DISSIPATION

Let us calculate the energy dissipation rate per the unit length in the y direction, averaged with respect to y. When we took all variables proportional to e^{iky}, we implicitly as-

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sumed that we consider the real parts of quantities. Then for the mean value of the product of the two quantities, g and h, we have

$$\frac{k}{2\pi}\int_0^{2\pi/k}\{\Re(ge^{iky})\}\{\Re(he^{iky})\}dy=\frac{1}{2}\Re(gh^*). \tag{67}$$

Using this result and Ampère's law we immediately obtain the energy dissipation rate averaged with respect to y:

$$\frac{d\mathcal{E}}{dt}=\frac{1}{2\sigma_0}\int_0^a dx\int_0^L|\mathbf{j}|^2\,dz=\frac{\eta}{2\mu}\int_0^a dx\int_0^L|\boldsymbol{\nabla}\times\mathbf{b}|^2\,dz, \tag{68}$$

where $\sigma_0=1/\mu\eta$ is the resistivity and $\mathbf{j}$ is the density of the electrical current. Dissipation becomes substantial only when large gradients in the x direction are built up in the vicinities of x_{A} and x_g. Note that x_g should be labeled by the subscript "n" because it is different for different harmonics with respect to z. We drop this subscript "n" for brevity. When large gradients are present, the y component of the magnetic field dominates other components in the vicinities of x_{A} and x_g. This observation enables us to use the approximation $|\boldsymbol{\nabla}\times\mathbf{b}|\approx|\partial b_y/\partial x|$.

To calculate b_y we use Eq. (4). The ratio of the second term on the right-hand side of this equation to the term on the left-hand side is of the order $\mathrm{R}^{-1/3}\ll 1$ even in the stationary dissipative layer embracing x_{A} in the steady oscillation state. These two terms are of the same order only in regions where the oscillation amplitude varies on the spatial scale of the order of $a\mathrm{R}^{-1/2}$. Such small spatial scales appear in the vicinity of x_g in the case where $\gamma\gg\Omega\mathrm{R}^{-1/3}$. However, as it was explained in the previous section, these small scales are built up at times when the amplitude of oscillation with the frequency ω_r is exponentially small, so it does not contribute significantly to energy dissipation. This analysis enables us to neglect the second term on the right-hand side of Eq. (4) when calculating b_y. Then, using Eq. (12) and carrying out the integration with respect to z, we arrive at

$$\frac{d\mathcal{E}}{dt}=\frac{\pi^2\rho V_{\mathrm{A}}^2\eta}{4L}\sum_{n=1}^{\infty}n^2\int_0^a\left|\frac{\partial}{\partial x}\int_0^t v_n(\tau)\,d\tau\right|^2dx. \tag{69}$$

We are only interested in the asymptotic behavior of this quantity for $t\Omega\gg 1$. To obtain the asymptotic expression for $\int_0^t v_n\,d\tau$, we use the same procedure as we used in Sec. VI to obtain Eq. (50). Then we retain only the resonant terms, because only these terms contribute into the energy dissipation. As a result we arrive at an expression coinciding with that obtained by the simple substitution for v_n of Eq. (50) with the third term in the curly brackets neglected.

Since we have assumed that the driver is in resonance only with the fundamental harmonic, the first term in the curly brackets in Eq. (50) is of the order δ_ω at any spatial position [see Eq. (53)] when $n>1$, and its contribution into $d\mathcal{E}/dt$ is negligible. Hence, for $n>1$,

$$\begin{aligned}\int_0^a&\left|\frac{\partial}{\partial x}\int_0^t v_n(\tau)\,d\tau\right|^2dx\\&\approx\frac{1}{4}\int_0^a\left\{\frac{\partial}{\partial x}\frac{k}{\rho\delta_\omega}\Re\left[A_g\int_0^t e^{-\gamma\tau-i\omega_r\tau}\,d\tau\right.\right.\\&\quad\times\int_0^{\tau\delta_\omega}\exp\left(\gamma s/\delta_\omega-\frac{1}{3}s^3\right)\\&\quad\times(\exp[i(\omega_r-V_{\mathrm{A}}nl)s/\delta_\omega]\\&\quad\left.\left.+\exp[i(\omega_r+V_{\mathrm{A}}nl)s/\delta_\omega])\,ds\right]\right\}^2dx.\end{aligned} \tag{70}$$

The expression in the square brackets is large only in the vicinity of the resonant position x_g, and it is large because of the resonance in the first exponent in the parentheses. The second exponent is nonresonant and can be neglected. Using integration by parts, we get rid of the integration with respect to s. Since the dominant contribution comes from the vicinity of x_g, we use the approximation (37). Making the substitution of the integration variable $x=x_g+\delta_{\mathrm{A}}\sigma$ with $\delta_{\mathrm{A}}=2\omega\delta_\omega/|\Delta|$, and neglecting derivatives of the equilibrium quantities in comparison with derivatives of quantities depending on σ, we obtain

$$\begin{aligned}\int_0^a&\left|\frac{\partial}{\partial x}\int_0^t v_n(\tau)\,d\tau\right|^2dx\\&\approx\frac{k^2}{4\rho^2\Delta^2\delta_{\mathrm{A}}^3}\int_{-\infty}^{\infty}\left\{A_g\left[e^{-\gamma t-i\omega_r t}\right.\right.\\&\quad\times\int_0^{t\delta_\omega}\tau\exp\left(i\sigma\tau\,\mathrm{sign}(\Delta)+\gamma\tau/\delta_\omega-\frac{1}{3}\tau^3\right)d\tau\\&\quad\left.-\int_0^{t\delta_\omega}\tau\exp\left(i\sigma\tau\,\mathrm{sign}(\Delta)-i\tau\omega_r/\delta_\omega-\frac{1}{3}\tau^3\right)d\tau\right]\\&\quad\left.+\mathrm{c.c.}\right\}^2d\sigma,\end{aligned} \tag{71}$$

where "c.c." denotes the complex conjugate quantity. It is straightforward to show, using integration by parts, that the ratio of the second integral in the square brackets to the first one is of the order of $\delta_\omega/\omega_r\ll 1$. Hence the second integral can be neglected. Now we square the expression in the curly brackets, and write the product of two integrals as a double integral with respect to τ and τ'. Then we change the order of integration and use the identity

$$\int_{-\infty}^{\infty}\exp[i\sigma(\tau-\tau')]\,d\sigma=2\pi\delta(\tau-\tau'), \tag{72}$$

where δ indicates the Dirac delta function. As a result we arrive at

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$$\int_0^a \left| \frac{\partial}{\partial x} \int_0^t v_n(\tau)\, d\tau \right|^2 dx \approx \frac{\pi k^2 |A_g|^2}{\rho^2 \Delta^2 \delta_{\mathrm{A}}^3} e^{-2\gamma t} \int_0^{t\delta_\omega} \tau^2 \exp\left(2\gamma\tau/\delta_\omega - \frac{2}{3}\tau^3 \right) d\tau. \tag{73}$$

All equilibrium quantities in this expression are calculated at $x = x_g$. Note that, although we did not average with respect to time, the right-hand side of this expression does not oscillate with the frequency ω_r. This property was noticed in numerical simulations,[10] and demonstrated by Wright and Allan[42] subsequently. Here we have proven the result analytically. The dependence on n is hidden in expressions for the equilibrium quantities, γ and A_g.

When $n=1$ the calculation is more complicated, because we have to take both terms in the curly brackets in Eq. (50) into account. As we have already explained, the dominant contribution into energy dissipation comes from the narrow dissipative layers embracing the two resonant positions, x_{A} and x_g. For $1 \ll t\Omega \ll \mathrm{R}^{1/3}$ the thickness of these layers can be substantially larger than δ_{A}, but it is always much smaller than a. If $|x_{\mathrm{A}} - x_g| \sim a$, the two dissipative layers do not overlap, and the integral over $[0,a]$ of the product of the two terms in Eq. (50) is very small and can be neglected. We consider a more general case where $|x_{\mathrm{A}} - x_g|$ can be much smaller than a. Then, using Eq. (50), we obtain for $n=1$ the expression similar to Eq. (70), however containing three terms on the right-hand side. The first term is obtained from the right-hand side of Eq. (70) by taking $n=1$. The second term is obtained from the right-hand side of Eq. (70) by taking $n=1$, $\gamma=0$, and substituting Ω and A_Ω for ω_r and A_g respectively. The third term comes from the product of the two terms on the right-hand side of Eq. (50), and it takes the form

$$2\int_0^a \left\{ \frac{\partial}{\partial x} \frac{k}{\rho\delta_\omega} \Re\left[A_g \int_0^t e^{-\gamma\tau - i\omega_r\tau} d\tau \times \int_0^{t\delta_\omega} \exp\left((\gamma + i\omega_r)s/\delta_\omega - \frac{1}{3}s^3 \right) \cos(V_{\mathrm{A}} l s/\delta_\omega)\, ds \right] \right\} \times \left\{ \frac{\partial}{\partial x} \frac{k}{\rho\delta_\omega} \Re\left[A_\Omega \int_0^t e^{-i\Omega\tau} d\tau \times \int_0^{t\delta_\omega} \exp\left(i\Omega s - \frac{1}{3}s^3 \right) \cos(V_{\mathrm{A}} l s/\delta_\omega)\, ds \right] \right\} dx. \tag{74}$$

The evaluation of the first term exactly coincides with that for $n>1$, and results in the expression on the right-hand side of Eq. (73). The evaluation of the second term is similar to that of the first term, and results in the expression obtained by substituting A_Ω for A_g and taking $\gamma=0$ in the expression on the right-hand side of Eq. (73). Let us now evaluate the expression (74). We assume that $|x_{\mathrm{A}} - x_g| \ll a$, because otherwise this expression is small and can be neglected. Then we write x_g as $x_g = x_{\mathrm{A}} + \zeta\delta_{\mathrm{A}}$, where ζ is a free parameter, and calculate all equilibrium quantities at $x = x_{\mathrm{A}}$. The procedure of evaluation of the expression (74) is similar to that which resulted in Eq. (73). First, using integration by parts, we get rid of integration with respect to s. Then we write the cosines as sums of two exponents and neglect the nonresonant ones. Then we neglect the derivatives of equilibrium quantities in comparison with derivatives of terms varying on the spatial scale δ_{A}. Now we make the substitution $x = x_{\mathrm{A}} + \delta_{\mathrm{A}}\sigma$, and integrate only over a narrow dissipative layer with the thickness of the order of δ_{A} embracing both x_{A} and x_g. This enables us to use the approximation (37) and write $\Omega - V_{\mathrm{A}} l = \delta_\omega \sigma \operatorname{sign}(\Delta)$, $\omega_r - V_{\mathrm{A}} l = \delta_\omega(\sigma - \zeta)\operatorname{sign}(\Delta)$. Then, neglecting small terms similar to the second term in the square brackets in Eq. (71), we rewrite the expression (74) as

$$\frac{k^2}{2\rho^2\Delta^2\delta_{\mathrm{A}}^3} \int_{-\infty}^{\infty} \left\{ A_\Omega e^{-i\omega t} \int_0^{t\delta_\omega} \tau \times \exp\left(i\sigma\tau \operatorname{sign}(\Delta) - \frac{1}{3}\tau^3 \right) d\tau + \text{c.c.} \right\} \times \left\{ A_g e^{-\gamma t - i\omega_r t} \int_0^{t\delta_\omega} \tau \exp\left(i(\sigma - \zeta)\tau \operatorname{sign}(\Delta) + \gamma\tau/\delta_\omega - \frac{1}{3}\tau^3 \right) d\tau + \text{c.c.} \right\} d\sigma. \tag{75}$$

Once again we write the product of the two integrals with respect to τ as a double integral with respect to τ and τ', change the order of integration, and use Eq. (72). Then, recalling expressions for the first and second terms, we eventually arrive at

$$\int_0^a \left| \frac{\partial}{\partial x} \int_0^t v_1(\tau)\, d\tau \right|^2 dx \approx \frac{\pi k^2}{2\rho^2\Delta^2\delta_A^3} \left\{ |A_\Omega|^2 (1 - e^{-2(t\delta_\omega)^3/3}) + 2|A_g|^2 e^{-2\gamma t} \int_0^{t\delta_\omega} \tau^2 \exp\left(2\gamma\tau/\delta_\omega - \frac{2}{3}\tau^3 \right) d\tau + 4\Re\left[A_\Omega A_g^* e^{-\gamma t + i(\Omega - \omega_r)t} \int_0^{t\delta_\omega} \tau^2 \times \exp\left(-i\zeta\tau \operatorname{sign}(\Delta) + \gamma\tau/\delta_\omega - \frac{2}{3}\tau^3 \right) d\tau \right] \right\}. \tag{76}$$

We see that the last term in the curly brackets describes the oscillations with the frequency $|\Omega - \omega_r|$, which is the beat between oscillations with the frequencies Ω and ω_r. This beat phenomenon was first observed by Poedts and Kerner[24] in the numerical simulation. These authors also suggested the explanation of this phenomenon as the beat between the driving frequency and the frequency of the global mode. Our analysis supports this explanation completely.

The case where $|x_{\mathrm{A}} - x_g| \sim a$ or, which is the same, $|\Omega - \omega_r| \sim \Omega$, corresponds to $\zeta \to \infty$. In this case the third term in the curly brackets on the right-hand side of Eq. (76) tends to zero and the beat disappears. When the driver is in resonance with the global mode ($\Omega = \omega_r$), the beat also disappears.

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In the steady state of oscillation both the right-hand side of Eq. (73) and the second and third terms in the curly brackets in Eq. (76) tend to zero. As a result we obtain for the energy dissipation rate

$$\frac{d\mathcal{E}}{dt}=\frac{\pi k^2 L\Omega|A_\Omega|^2}{8\rho|\Delta|}, \tag{77}$$

where all equilibrium quantities are calculated at $x=x_{\mathrm{A}}$. Note that the energy dissipation rate is independent of η, which once again is a well-known result.

Let us now estimate the characteristic time necessary that $d\mathcal{E}/dt$ reaches its stationary value. First we consider the nonresonant driving where $|\Omega-\omega_r|\gg\gamma$. When $\gamma\lesssim\Omega\mathrm{R}^{-1/3}$, the upper limits of integration in Eqs. (73) and (76) can be substituted by infinity for $t\gtrsim\gamma^{-1}$. Then it becomes obvious that the characteristic transitional time for $d\mathcal{E}/dt$ is γ^{-1}. When $\gamma\gg\Omega\mathrm{R}^{-1/3}$, the analysis, similar to that used to study Eq. (57), shows that the right-hand side of Eq. (73) and the second and third term in the curly brackets on the right-hand side of Eq. (76) decay on the time scale of the order of $\Omega^{-1}\mathrm{R}^{1/3}$. Hence, we conclude that the characteristic transitional time for $d\mathcal{E}/dt$ is t_{tr} given by Eq. (55).

Now we consider the resonant driving. For the sake of simplicity we take $\Omega=\omega_r$; however, the analysis remains valid also for $|\Omega-\omega_r|\lesssim\gamma$. The amplitude of the fundamental harmonic is now larger than the amplitudes of the overtones by a factor of the order of L/a. This implies that the terms with $n>1$ in the sum on the right-hand side of Eq. (69) can be neglected in comparison with the first term. Then, using Eq. (76), the relation $A_g\approx -A_\Omega$, and integration by parts, we obtain

$$\frac{d\mathcal{E}}{dt}=\frac{\pi k^2 L\Omega|A_\Omega|^2}{8\rho|\Delta|}\left\{(1-e^{-\gamma t})^2-\frac{2\gamma}{\delta_\omega}\int_0^{t\delta_\omega}(1-e^{-\gamma(t-\tau/\delta_\omega)})\right.$$
$$\left.\times\exp\left(-\gamma(t-\tau/\delta_\omega)-\frac{2}{3}\tau^3\right)d\tau\right\}. \tag{78}$$

Once again it is obvious that the characteristic time is γ^{-1} when $\gamma\lesssim\Omega\mathrm{R}^{-1/3}$. Let us study the case where $\gamma\gg\Omega\mathrm{R}^{-1/3}$. For $t\delta_\omega\ll 1$ we can take $e^{-2\tau^3/3}\approx 1$ in the integral in Eq. (78). Then the integral is easily calculated, and we find that the second term in the curly brackets in Eq. (78) is approximately equal to unity for $t\gg\gamma^{-1}$. This consideration shows that the transitional time cannot be much smaller than $\delta_\omega^{-1}\sim\Omega\mathrm{R}^{-1/3}$. If we neglect the exponent in the parentheses in the integrand, we only increase the second term. On the other hand, after this neglect it becomes similar to the absolute value of the expression (57) at $\sigma=0$. Then we can use the analysis of this expression carried out in the previous section, and conclude that the characteristic decay time for the second term is $\Omega^{-1}\mathrm{R}^{1/3}$. Hence, the characteristic time necessary so that $d\mathcal{E}/dt$ reaches its stationary value is always t_{tr}, no matter if the driving is resonant or nonresonant. In particular, this time is $\Omega^{-1}\mathrm{R}^{1/3}$ when $\gamma\gtrsim\Omega\mathrm{R}^{-1/3}$. Hence, we have not found the effect obtained by Poedts and Kerner.[24] On the basis of numerical modeling these authors claimed that the characteristic time is proportional to $\mathrm{R}^{1/5}$ in the case of resonant driving. However, the difference between our result and that by Poedts and Kerner[24] is not surprising at all, because the setting of the problem in the present article differs very much from that in the paper by Poedts and Kerner.[24]

VIII. SUMMARY AND CONCLUSIONS

In this article we have addressed the problem of nonstationary harmonically driven oscillations of a one-dimensional magnetic cavity in a cold resistive plasma. We have studied only foot point driving, polarized in the inhomogeneous direction. We have considered small-amplitude oscillations and used the linear description. The analysis has been based on the two main assumptions: (i) the magnetic Reynolds number R is large and (ii) the cavity width in the direction perpendicular to the equilibrium magnetic field, a, is much smaller than the cavity length in the direction of the magnetic field, L. These two assumptions have greatly simplified the analysis. In particular, the second assumption has enabled us to neglect the variation of the magnetic pressure in the direction perpendicular to the equilibrium magnetic field. As a result we have managed to obtain the solution describing the temporal evolution of the magnetic pressure and the velocity explicitly. Using this solution we have studied the transition to the steady state of driven oscillations, and calculated the energy dissipation rate in the cavity.

On the basis of our analysis we have made the following conclusions. One should distinguish between the global motion of the cavity and the local motion. The global motion corresponds to the magnetic pressure oscillation, and the oscillation of the magnetic field lines in the inhomogeneous direction x, with the exception of field lines in the vicinities of the resonant positions. For $t\Omega\gg 1$ (t is the time and Ω the driving frequency) the global motion of the cavity is exactly the same as that of a damped oscillator. After the transitional time of the order of $\gamma^{-1}\sim L/a\Omega$ (γ is the decrement of the fundamental global mode) it attains the steady state of harmonic oscillation with frequency Ω. When the driving is nonresonant ($|\Omega-\omega_r|\gg\gamma$, where ω_r is the real frequency of the fundamental global mode), the amplitude of the global motion is of the order of the driving amplitude. When the driving is quasi-resonant ($|\Omega-\omega_r|\lesssim\gamma$), the amplitude is larger than the driving amplitude by factor of the order of $\Omega/\gamma\sim L/a$.

The local motion is the motion in the y direction, which is the direction perpendicular to the equilibrium magnetic field and to the direction of inhomogeneity, and also the motion in the x direction in the vicinities of resonances. We have assumed that Ω is in the Alfvén continuum of the fundamental harmonic with respect to z, so there is a resonant position x_{A} where the local Alfvén frequency matches Ω. In addition, there is the countable set of resonant positions x_{gn}, where the local Alfvén frequency matches either the frequency of the fundamental global mode ($n=1$), or the frequency of the global modes corresponding to overtones ($n>1$).

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The motion of a magnetic line in the y direction is a superposition of the oscillation with the frequency Ω, and the oscillation with the local Alfvén frequency. Oscillations at the local Alfvén frequencies decay on the phase-mixing time scale $\Omega^{-1}\mathrm{R}^{1/3}$, and after that all the magnetic field lines oscillate with the frequency Ω. Far away from the resonant positions the oscillation amplitudes are of the order of the driving amplitude. In a thin dissipative layer with the thickness $\delta_\mathrm{A}\sim a\mathrm{R}^{-1/3}$ the oscillation amplitude is larger than the driving amplitude by a factor of the order of $\mathrm{R}^{1/3}$.

Since the fundamental global mode dominates the global modes of the overtones, and it also has the largest damping time ($\gamma=\gamma_1<\gamma_n$ for $n>1$), in what follows we discuss only the motion in the vicinity of $x_g=x_{g1}$. The motion in the vicinity of x_g depends strongly on the ratio between the decrement of the global mode γ, and the inverse phase-mixing time $\Omega\mathrm{R}^{-1/3}$. When $\gamma\lesssim\Omega\mathrm{R}^{-1/3}$, the decay of the global motion with frequency ω_r is slow enough to make possible the formation of a quasi-stationary dissipative layer embracing x_g. For $t\sim\Omega^{-1}\mathrm{R}^{1/3}$ the thickness of this layer, and the amplitude of the oscillation with frequency ω_r inside it, are of the same order of magnitude as those in the vicinity of x_A. Then the motion in this layer decays slowly on the time scale γ^{-1}.

When $\gamma\gg\Omega\mathrm{R}^{-1/3}$, the amplitude of the oscillation with the frequency ω_r in the vicinity of x_g attains its maximum value (which is larger than the driving amplitude by a factor of the order of $\Omega/\gamma\sim L/a\ll\mathrm{R}^{1/3}$) at time $t_m\sim\gamma^{-1}\ln(\gamma\mathrm{R}^{1/3}/\Omega)$. Then it slowly decays on the phase-mixing time scale $\Omega^{-1}\mathrm{R}^{1/3}$.

The motion of a magnetic field line in the x direction also consists of a superposition of an oscillation with frequency Ω, and an oscillation with the local Alfvén frequency. However, in contrast to the motion in the y direction, the oscillation with frequency Ω dominates the oscillation with the local Alfvén frequency everywhere, except in the vicinity of x_g, after a time of the order of γ^{-1}, even in the case where $\gamma\gg\Omega\mathrm{R}^{-1/3}$. This fact enables us to claim that the global motion of the cavity, which is the coherent motion in the x direction, attains its steady state after a time of the order of γ^{-1}.

In the dissipative layer embracing x_A the motion in the x direction has an amplitude of the order of the driving amplitude multiplied by $\ln\mathrm{R}$. When $\gamma\lesssim\Omega\mathrm{R}^{-1/3}$, the amplitude of the oscillation in the x direction with frequency ω_r in the vicinity of x_g is of the same order of magnitude as in the vicinity of x_A. On the other hand, when $\gamma\gg\Omega\mathrm{R}^{-1/3}$, this amplitude always remains smaller or of the order of the driving amplitude. These results show that the characteristic time for the transition of the local motion to the steady state of oscillation is $t_\mathrm{tr}=\max(\gamma^{-1},\Omega^{-1}\mathrm{R}^{1/3})$.

The energy dissipation rate attains its stationary value also after a time of the order of t_tr. When $|\Omega-\omega_r|\sim\Omega$, or $|\Omega-\omega_r|\lesssim\gamma$ (the case of the quasi-resonance), it monotonically increases to its stationary value. However, when $\gamma\ll|\Omega-\omega_r|\ll\Omega$, it oscillates with the beat frequency $|\Omega-\omega_r|$. This oscillation is caused by the overlapping of the two dissipative layers, one embracing x_A and the other x_g. This phenomenon was first found in the numerical simulation by Poedts and Kerner.[24]

Poedts and Kerner[24] have found that the transitional time to the steady state of energy dissipation is proportional to $\mathrm{R}^{1/5}$ in the case of resonant driving ($\Omega=\omega_r$). We have found in our analysis that this time is always proportional to $\mathrm{R}^{1/3}$, no matter whether the driving is resonant or nonresonant. However, this difference is not surprising at all. Poedts and Kerner[24] considered lateral driving in cylindrical geometry. In addition, the wavelength in the direction of the cylinder axis was of the order of the cylinder radius in their numerical simulation, so the long-wavelength approximation is not applicable to their study. Hence, the setting of the problem in the present article differs very much from that in the article by Poedts and Kerner.[24]

The main conclusion that we make on the basis of our analysis is that, in general, there are two different transitional times in the problem of driven oscillations of a magnetic cavity. The first transitional time is the time necessary for the global motion of the cavity, which is the coherent motion in the direction of the inhomogeneity, to attain a steady state of oscillation. This time is of the order of γ^{-1}. The second transitional time is the time necessary for the motion in the y direction, and the energy dissipation rate to attain their steady states. This time is of the order of t_tr.

Note that in practically all applications $\gamma\gg\Omega\mathrm{R}^{-1/3}$. In this respect let us consider one example from solar physics. The ratio of the length of a coronal magnetic loop to its radius is always smaller than 100. This implies the estimate $\gamma/\Omega\gtrsim 0.01$. If we use the formulas based on classical Coulomb collisions, we obtain $\mathrm{R}\gtrsim 10^{12}$. Assume that one end of the loop has started to be harmonically driven with a period of 1 min. Then the transition to the steady state of the global oscillation of the loop, which can be observed, will take about 1 h or less. On the other hand, the energy dissipation rate in the loop attains its stationary value only after a few days.

To make the main results obtained in this article more accessible, we collected them in the following table:

		non-resonant driving		quasi-resonant driving	
		$\gamma\lesssim\Omega R^{-1/3}$	$\gamma\gg\Omega R^{-1/3}$	$\gamma\lesssim\Omega R^{-1/3}$	$\gamma\gg\Omega R^{-1/3}$
Global or coherent motion	Amplitude	$\sim f_0$	$\sim f_0$	$\sim Lf_0/a$	$\sim Lf_0/a$
	Transitional time to the steady state	$\sim\gamma^{-1}$	$\sim\gamma^{-1}$	$\sim\gamma^{-1}$	$\sim\gamma^{-1}$
Dissipative layer embracing x_A	Thickness	$\delta_A\sim aR^{-1/3}$	$\delta_A\sim aR^{-1/3}$	$\delta_A\sim aR^{-1/3}$	$\delta_A\sim aR^{-1/3}$
	Amplitude of v	$\sim f_0R^{1/3}$	$\sim f_0R^{1/3}$	$\sim f_0(L/a)R^{1/3}$	$\sim f_0(L/a)R^{1/3}$
	Transitional time to the steady state	$\sim\gamma^{-1}$	$\sim\Omega^{-1}R^{1/3}$	$\sim\gamma^{-1}$	$\sim\Omega^{-1}R^{1/3}$
Dissipative layer embracing x_{gn}	Thickness	$\delta_A\sim aR^{-1/3}$	$\delta_A\sim aR^{-1/3}$	$\delta_A\sim aR^{-1/3}$ ($n>1$)	$\delta_A\sim aR^{-1/3}$ ($n>1$)
	Amplitude of v	$\sim f_0R^{1/3}$	$\sim Lf_0/a$	$\sim f_0R^{1/3}$ ($n>1$)	$\sim Lf_0/a$ ($n>1$)
	Transitional time to the steady state	$\sim\gamma^{-1}$	$\sim\Omega^{-1}R^{1/3}$	$\sim\gamma^{-1}$	$\sim\Omega^{-1}R^{1/3}$

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APPENDIX A: CALCULATION OF $\partial P_n/\partial x$ AT THE INHOMOGENEOUS LAYER BOUNDARIES

In this appendix we calculate $\partial P_n/\partial x$ at $x=0,a$. In regions I and II P is determined by Eq. (11) with $d\rho/dx=0$. The substitution of Eq. (12) in this equation results in the equation for P_n. P_n vanishes as $|x|\to\infty$. Since the cavity is at rest for $t\leqslant 0$, $P_n=\partial P_n/\partial t=0$ at $t=0$.

Let us first solve the equation for P_n in region II. Substituting $P_n=Q+P_n(a)e^{-\kappa_n x'}$ with $x'=x-a$ into the equation for P_n, and dropping the prime, we obtain

$$\frac{\partial^2 Q}{\partial t^2}-V_A^2\frac{\partial^2 Q}{\partial x^2}+V_A^2\kappa_n^2 Q=-\frac{d^2P_n(a)}{dt^2}e^{-\kappa_n x}. \tag{A1}$$

Q satisfies the boundary conditions $Q=0$ at $x=0$, $Q\to 0$ as $x\to\infty$, which enables us to expand Q into the sine-Fourier integral:

$$Q(x)=\frac{2}{\pi}\int_0^\infty Q_\chi \sin(x\chi)\,d\chi,\qquad Q_\chi=\int_0^\infty Q(x)\sin(x\chi)\,dx. \tag{A2}$$

Then we obtain from Eq. (A1)

$$\frac{d^2Q_\chi}{dt^2}+V_A^2(\kappa_n^2+\chi^2)Q_\chi=-\frac{\chi}{\kappa_n^2+\chi^2}\frac{d^2P_n(a)}{dt^2}. \tag{A3}$$

The solution to this equation satisfying the initial conditions $Q_\chi=dQ_\chi/dt=0$ at $t=0$ is

$$Q_\chi=-\int_0^t\frac{d^2P_n(a)}{d\tau^2}\,\frac{\chi\sin[V_A(\kappa_n^2+\chi^2)^{1/2}(t-\tau)]}{V_A(\kappa_n^2+\chi^2)^{3/2}}\,d\tau. \tag{A4}$$

Using Eq. (A2) and the relation between P_n and Q, we obtain from Eq. (A4)

$$\left.\frac{\partial P_n}{\partial x}\right|_{x=a}=-\kappa_n P_n(a)-\frac{2}{\pi V_A}\int_0^t\frac{d^2P_n(a)}{d\tau^2}\,d\tau\times\int_0^\infty\frac{\chi^2\sin[V_A(\kappa_n^2+\chi^2)^{1/2}(t-\tau)]}{(\kappa_n^2+\chi^2)^{3/2}}\,d\chi. \tag{A5}$$

With the aid of the formula[43]

$$\int_0^\infty\frac{\sin[c(x^2+y^2)^{1/2}]}{(x^2+y^2)^{1/2}}\cos(bx)\,dx=\frac{\pi}{2}H(c-b)J_0[y(c^2-b^2)^{1/2}]$$

with H the Heaviside function, we obtain

$$\int_0^\infty\frac{\sin[V_A(\kappa_n^2+\chi^2)^{1/2}(t-\tau)]}{(\kappa_n^2+\chi^2)^{1/2}}\,d\chi=\frac{\pi}{2}J_0[V_A\kappa_n(t-\tau)]. \tag{A6}$$

Using this formula and integration by parts we eventually transform Eq. (A5) into Eq. (14) with $j=2$. Equation (14) with $j=1$ is obtained from Eq. (14) with $j=2$ by taking $a=0$ and substituting $-x$ for x.

APPENDIX B: INVESTIGATION OF EQ. (57)

Let us find out when the amplitude of oscillations, described by Eq. (57), attains its maximum value at $\sigma=0$. We consider the case where $\gamma\gg\delta_\omega$. Discarding $e^{-i\omega_\tau t}$ and the constant multiplier, we write the essential part of Eq. (57), determining the dependence of the oscillation amplitude on time, as $\mathcal{F}(\xi)=e^{-\alpha\xi}F(-i\alpha,\xi)$, where $\alpha=\gamma/\delta_\omega\gg 1$ and $\xi=t\delta_\omega$. The function $\mathcal{F}(\xi)$ takes its maximum value at ξ determined by $d\mathcal{F}/d\xi=0$. Differentiating $\mathcal{F}(\xi)$ and using integration by parts, we write this condition as

$$\mathcal{G}(\xi)\equiv\int_0^\xi s^2\exp\left(\alpha s-\frac{1}{3}s^3\right)ds=1. \tag{B1}$$

Since $\mathcal{G}(0)=0$, $\mathcal{G}(\infty)>\int_0^\infty s^2\exp(-s^3/3)ds=1$, and $d\mathcal{G}/d\xi>0$, this equation has exactly one positive solution. Let us try to find the solution satisfying the condition $\xi\ll 1$. This condition enables us to neglect $s^3/3$ in the exponent in Eq. (B1), and rewrite it in the approximate form

$$(\alpha^2\xi^2-2\alpha\xi+2)e^{\alpha\xi}=2\alpha^3. \tag{B2}$$

Obviously, ξ can satisfy this equation only if $\alpha\xi\gg 1$, so we can neglect the second and third term in the brackets. Then, taking the logarithm of both sides of Eq. (B2), we obtain

$$\alpha\xi=\alpha\xi_m\approx 3\ln\alpha-2\ln\ln\alpha. \tag{B3}$$

Since ξ_m satisfies the conditions $\xi_m\ll 1$ and $\alpha\xi_m\gg 1$, it is the approximate solution to Eq. (B1). The second term on the right-hand side of Eq. (B3) is much smaller than the first one and can be neglected. Then, returning to the initial variables, we obtain that the maximum amplitude in Eq. (57) is attained at $t=t_m\approx 3\gamma^{-1}\ln(\gamma/\delta_\omega)$. Once again, neglecting $s^3/3$ in the exponent in the expression for $\mathcal{F}(\xi)$, we immediately obtain

$$\mathcal{F}(\xi_m)\approx\alpha^{-1}=\delta_\omega/\gamma. \tag{B4}$$

Note that $d\mathcal{F}/d\xi=-e^{-\alpha\xi}[\mathcal{G}(\xi)-1]$, so $d\mathcal{F}/d\xi<0$ for $\xi>\xi_m$, i.e., $\mathcal{F}(\xi)$ monotonically decreases for $\xi>\xi_m$.

Using integration by parts we obtain

$$\mathcal{F}(\xi)=\frac{1}{\alpha}e^{-\xi^3/3}-\frac{1}{\alpha}e^{-\alpha\xi}+\frac{1}{\alpha}e^{-\alpha\xi}\times\int_0^\xi s^2\exp\left(\alpha s-\frac{1}{3}s^3\right)ds. \tag{B5}$$

Let us take $\xi<\alpha^{1/2}$. Then the second term on the right-hand side of Eq. (B7) is much smaller than the first one and can be neglected. If we substitute ξ^2 for s^2 in the integrand in the last term, we increase this term, and obtain $\xi^2\mathcal{F}(\xi)/\alpha<\mathcal{F}(\xi)$. This implies that the last term also can be neglected. Hence, for $\xi<\alpha^{1/2}$, we obtain the approximate expression

$$\mathcal{F}(\xi)\approx\alpha^{-1}e^{-\xi^3/3}. \tag{B6}$$

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