COMMENTS

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Comment on "Validity of the field line resonance expansion" [Phys. Fluids B 4, 2713 (1992)]

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In a recent paper Hansen and Goertz (hereafter HG) [Phys. Fluids B **4**, 2713 (1992)] considered the coupling between fast and Alfvén modes in a cold plasma containing a uniform magnetic field ($B_0\hat{\mathbf{x}}$) extending between two perfectly reflecting plane boundaries at constant x. The equilibrium medium is invariant in only one direction ($\hat{\mathbf{y}}$), and (importantly) the density may vary along the equilibrium field lines, $\rho_0(x,z)$. HG sought solutions of the coupled governing PDEs (partial differential equations) for linear perturbations of the form $\exp i(k_y y-\omega t)$. The solution has been studied previously [Planet. Space Sci. **22**, 483 (1974); J. Geophys. Res. **79**, 1024 (1974)] in the case when ρ_0 does not vary along the background field lines, when each Fourier mode in x decouples from the others and may be considered separately—reducing the problem to an ODE (ordinary differential equation). In this case a logarithmic singularity exists at the resonant field line where $\omega^2=k_x^2V_A^2(z)$, V_A being the Alfvén speed ($V_A^2=B_0^2/4\pi\rho_0$). HG claim the introduction of density variation along the equilibrium field causes the modes in x to become coupled resulting in the singular ODE solution becoming a *nonsingular* solution in the PDE case. If this conclusion is true it is of great importance for researchers in many areas such as solar corona and laboratory plasma heating, and magnetospheric pulsations. Indeed, it suggests that a large portion of the existing literature in these fields is wrong. Clearly it is important to decide whether the calculation of HG is correct or not. In this Comment the equations they set up are analyzed and are solved in a different fashion to HG. The solution found is different from that of HG and in agreement with the existing body of literature. Some sources of error in HG's analysis are pointed out.

Hansen and Goertz[1] (HG) show how the governing cold magnetohydrodynamic (MHD) equations may be written [HG Eqs. (13)–(15)]:

$$L\xi_z=-\frac{\partial b}{\partial z};\quad L\xi_y=-ik_yb;\quad b=ik_y\xi_y+\frac{\partial \xi_z}{\partial z}. \tag{1}$$

Here $b=-b_x/B_0$, where b_x is the compressional magnetic field perturbation. The operator L is defined as $L=\partial^2/\partial x^2+\omega^2/V_A^2(x,z)$. The plasma displacement (ξ) is required to vanish at the planes $x=x_1,x_2$. Then the operator L has eigenmodes Q_n with eigenfrequency ω_n:

$$\frac{\partial^2 Q_n(x,z)}{\partial x^2}=-\frac{\omega_n^2(z)}{V_A^2(x,z)}\,Q_n(x,z). \tag{2}$$

Moreover, the modes are orthonormal when weighted with V_A^{-2},

$$\int_{x_1}^{x_2}\frac{Q_nQ_p}{V_A^2}\,dx\equiv\left\langle\frac{Q_nQ_p}{V_A^2}\right\rangle=\delta_{np}\quad \forall z, \tag{3}$$

and complete, so that we may write the dependence of the plasma displacements on x and z as a sum over the eigenmodes:

$$\xi_y(x,z)=\sum_n \mu_n(z)Q_n(x,z);$$

$$\xi_z(x,z)=\sum_n \lambda_n(z)Q_n(x,z). \tag{4}$$

Eliminating b from the governing equations (1), substituting for ξ with (4), and employing (2) we find two coupled equations for the eigenmode coefficients μ_n and λ_n:

$$\sum_n\frac{\mu_n[\omega^2-\omega_n^2(z)]Q_n}{V_A^2}=\sum_n\{k_y^2\mu_nQ_n-ik_y(\lambda_n'Q_n+\lambda_nQ_n')\}, \tag{5a}$$

$$\sum_n\frac{\lambda_n[\omega^2-\omega_n^2(z)]Q_n}{V_A^2}+\sum_n\{ik_y(\mu_n'Q_n+\mu_nQ_n')+\lambda_nQ_n''+2\lambda_n'Q_n'+\lambda_n''Q_n\}=0. \tag{5b}$$

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Here the prime denotes differentiation with respect to z. Thus far we have restated the problem formulated by HG up to equation (HG20), and agree with this formulation. HG now go on to consider the solution of the equations near the allegedly resonant/singular field line z_0 where the mth eigenmode is resonant: $\omega^2=\omega_m^2(z_0)$. They find that no singular solution exists.

An identical system to that described by the above equations was studied simultaneously to, and independently of, HG by Thompson and Wright[2] (hereafter TW). They find that a logarithmically singular solution does exist, in contrast to the findings of HG. Specifically, TW find that (in the notation of HG) the leading-order coefficients have the following variations,

$$\mu_m\sim 1/y;\quad \lambda_m\sim \ln y;\quad \mu_n \quad\text{and}\quad \lambda_n\sim\text{nonsingular},\quad n\neq m. \tag{6}$$

Thus only the resonant mode is singular. Note that we have introduced the parameter $y\equiv z-z_0$ as in HG—this should not be confused with the invariant direction $\hat{\mathbf{y}}$. (We have also allowed for the fact that TW use the eigenfunctions at $y=0$—see below.) Although HG did not find the ordering in (6), we shall now demonstrate how this may be deduced quite easily from the framework established above taken from HG.

Following the calculation of HG we assume a Frobenius series of the form

$$\mu_n=(A_1^n/y)+B_1^n\ln y+C_1^n+D_1^n y\ln y+F_1^n y+E_1^n y^2\ln y+G_1^n y^2+O(y^3,y^3\ln y), \tag{7}$$

$$\lambda_n=B_2^n\ln y+C_2^n+D_2^n y\ln y+F_2^n y+E_2^n y^2\ln y+G_2^n y^2+O(y^3,y^3\ln y).$$

TW actually prove that the most singular term is of order y^{-1} in ξ_y, so we may expect the above series to be a suitable expansion. If we are to derive the orderings of TW given in Eq. (6) we need to show that A_1^m and B_2^m may be nonzero, while all of the coefficients A_1^n and $B_{1,2}^n$ for $n\neq m$ must be zero.

[Readers who may be interested in looking through the analysis of HG should note that we have included extra terms in our series compared to (HG31). These terms will contribute to (HG33c) and (HG34d), and so must be included—although their omission does not affect the arguments put forward here. Also (HG33c) should include terms of order y^0 from HG's functions V_1, and (HG34d) should include similar terms from V_2. The term C_2 on the l h s of (HG34d) should be omitted.]

We begin by multiplying (5b) by $Q_p(x,z)/V_A^2$ and integrate between the boundaries in x at fixed (but arbitrary) z. Here Q_p represents any eigenmode. Recalling the orthogonality property (3) we may equate the coefficients of terms in y^{-2} to give

$$ik_yA_1^p+B_2^p=0\quad \forall p. \tag{8}$$

To proceed further we need to expand the eigenfrequencies in z also [cf. (HG32)]:

$$\omega_n^2(z)=\omega_n^2(z_0)+\gamma^n y+\cdots;\quad \gamma^n=\frac{d\omega_n^2}{dz},\quad \forall n. \tag{9}$$

(Recall $y\equiv z-z_0$.) Substitute (9) into (5a) multiplied by Q_p and integrate in x to give

$$\mu_p[\omega^2-\omega_p^2(z_0)-\gamma^p y+\cdots]=\sum_n\{(k_y^2\mu_n-ik_y\lambda_n')\langle Q_pQ_n\rangle-ik_y\lambda_n\langle Q_pQ_n'\rangle\}. \tag{10}$$

Substituting the Frobenius expansion (7) into (10) and collecting coefficients of y^{-1} terms gives [after employing (8)]

$$A_1^p[\omega^2-\omega_p^2(z_0)]=0; \tag{11}$$

i.e., the resonant coefficient [for which $\omega^2=\omega_m^2(z_0)$] may be arbitrary, but all nonresonant coefficients have $A_1^p=0$, $p\neq m$. In conjunction with (8) this result requires

$$B_2^m=-ik_yA_1^m;\quad B_2^p=0,\quad p\neq m. \tag{12}$$

Thus B_2^n may only be nonzero for the resonant (m) mode, and is zero for all nonresonant coefficients.

Multiplying Eq. (5b) by Q_p and integrating in x, again at arbitrary fixed z, yields

$$\lambda_p[\omega^2-\omega_p^2(z_0)-\gamma^p y+\cdots]=-\sum_n\{(ik_y\mu_n'+\lambda_n'')\langle Q_pQ_n\rangle+(ik_y\mu_n+2\lambda_n')\langle Q_pQ_n'\rangle+\lambda_n\langle Q_pQ_n''\rangle\}. \tag{13}$$

Substituting the series expansions (7) and (9) into (13), collecting the coefficients of y^{-1} terms and combining with the $\ln y$ coefficients from (10) we find

$$B_1^p[\omega^2-\omega_p^2(z_0)]=0, \tag{14}$$

where we have also used the result (8). Once more we find that the resonant coefficient B_1^m may be nonzero, but the nonresonant coefficients must be zero. The results (11), (12), and (14) represent the same ordering in (6) found by TW. Thus we find that the equations presented by HG *can* admit singular solutions, in contradiction to the findings of the more lengthy analysis they present.

The question of the discrepancy between the analysis of HG and our analysis above of their equations must be addressed. We believe that their calculation contains a serious error. To elucidate the source of the error, we first consider the solution given by TW. In the terminology of HG, the singular solution given by TW [their Eqs. (26)] is

$$\xi_y=(A_1/y)Q_m(x,z_0)-A_1(\langle\cdots\rangle/\langle\cdots\rangle)Q_m(x,z_0)+\tfrac{1}{2}k_y^2A_1y\ln yQ_m(x,z_0)+O(y,y^2\ln y); \tag{15a}$$

$$\xi_z=-ik_yA_1\ln yQ_m(x,z_0)+O(y,y^2\ln y). \tag{15b}$$

Note that this expansion is in terms of $Q_n(x,z_0)$, rather than the $Q_n(x,z)$ used by HG. The terms $\langle\cdots\rangle$ represent constants determined by structure of the equilibrium model. Since both pairs of authors are considering boundary conditions such that the solutions vanish at $x=x_1$,

$x=x_2$ *where* x_1, x_2 *are independent of* z, clearly either is an orthogonal and complete set that is suitable for expansions of the form (4). TW's solution may of course be expressed in terms of $Q_n(x,z)$. This may most simply be obtained from (15), by expanding $Q_n(x,z_0)$ as a Taylor series about an arbitrary z, viz.,

$$Q_n(x,z_0)=Q_n(x,z)-yQ_n'(x,z)+\cdots$$
$$\equiv Q_n(x,z)-y\sum_p \left\langle \frac{Q_n'Q_p}{V_A^2}\right\rangle Q_p(x,z)+\cdots .$$

Using this yields

$$\xi_y=\frac{A_1}{y}Q_m(x,z)+\cdots+\tfrac{1}{2}k_y^2A_1y\ln yQ_m(x,z)$$
$$-\tfrac{1}{2}k_y^2A_1y^2\ln y\sum_n \left\langle \frac{Q_m'Q_n}{V_A^2}\right\rangle Q_n(x,z)+\cdots; \quad (16a)$$
$$\xi_z=-ik_yA_1\ln yQ_m(x,z)$$
$$+ik_yA_1y\ln y\sum_n \left\langle \frac{Q_m'Q_n}{V_A^2}\right\rangle Q_n(x,z)+\cdots . \quad (16b)$$

Note that the solution we began to develop above, using Eqs. (11), (12), and (14), is consistent with the result quoted in (6). Indeed one could continue the procedure started above and reproduce (6) in full. However, the point we wish to emphasize is that in the expansion (16) in $Q_n(x,z)$ there are clearly terms involving nonresonant Q_n that are proportional to $y^r\ln y(r>0)$. These terms are not themselves singular, because they vanish as $y\to 0$, but their rth and higher derivatives will be. This point, overlooked by HG, is their principal error. Thus they assume "by hypothesis" (without attempting *a posteriori* justification) that their functions V_1, V_2 on the RHS of equations (HG30) are nonsingular. This is equivalent to saying that, setting $p=m$, the nonresonant ($n\neq m$) terms in our equations (10) and (13) are nonsingular. Yet those terms include first and second derivatives of λ_n and therefore, by (16b), *will* be singular in general. This is the fundamental error committed by HG. On the grounds of (16) we expect that to leading order $V_1\sim\ln y$ and $V_2\sim y^{-1}$ in clear contradiction to HG's hypothesis that these functions are nonsingular.

If HG had taken these terms into account, and allowing for the other small errors we have noted above, they should have been able to conclude that there is a consistent singular solution of their set of equations in which the resonant term is singular and the nonresonant terms are nonsingular. This would have been in accord with the assumed ordering of Chen and Cowley[3] and the detailed calculation of TW. Instead HG reached the opposite conclusion, for the reasons we have indicated, and deduced that no such singular solution exists; but their conclusion is erroneous.

Finally it might be noted that the resonant singularity in the PDEs system survives even when more general magnetic field geometries are considered, as has recently been shown.[4]

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[1] P. J. Hansen and C. K. Goertz, Phys. Fluids B **4**, 2713 (1992).
[2] M. J. Thompson and A. N. Wright, J. Geophys. Res. **98**, 15541 (1993).
[3] L. Chen and S. C. Cowley, Geophys. Res. Lett. **16**, 895 (1989).
[4] A. N. Wright and M. J. Thompson, Phys. Plasmas **1**, 691 (1994).