Analytical treatment of Alfvén resonances and singularities in nonuniform magnetoplasmas

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The linear resonant excitation of Alfvén waves in a cold plasma permeated by a nonuniform magnetic field is considered. The equilibrium magnetic field is irrotational and possesses an invariant coordinate perpendicular to the direction of the field. By solving for the coefficients in a Generalized Frobenius Series the regular and singular solutions may be generated. The singular solution is logarithmic and produces a net absorption of energy at the resonant field line. The efficiency of coupling between the fast mode and the resonant Alfvén mode is determined by the following overlap integral along the resonant field line, $\int (\xi_{\beta r} b_{\gamma 0}/h_\beta)\,dl$; $\xi_{\beta r}$ is the resonant eigenfunction, $b_{\gamma 0}$ is the compressional/parallel magnetic field perturbation, and h_β is proportional to the separation of background lines of force in the invariant direction. The amplitude of the singular solution is proportional to this integral, while the rate of energy absorption at the resonance is proportional to its square. It is also shown how the analytical solution at the resonance may be used to avoid problems encountered in numerical solutions.

I. INTRODUCTION

The coupling of different MHD (magnetohydrodynamic) wave modes in inhomogeneous media is a fundamental process of interest to all plasma physicists and is important for an understanding of solar, magnetospheric, and laboratory plasmas. The slow, Alfvén and fast modes may all couple with one another. It is the coupling of the fast and Alfvén mode which has received most attention in the literature, and is the problem which we address in this paper.

In both solar and laboratory plasmas the absorption of fast mode energy by spatially localized Alfvén waves is part of a mechanism for heating the plasma. Thus wave coupling can help achieve the high temperatures required for laboratory fusion, and can aid our understanding of the unusually high temperatures found in the solar corona. In a magnetospheric context the heating aspect of the coupling process is not so important, and attention is focused upon the structure of the Alfvén waves that may be excited from a fast mode. Indeed, it is thought that the properties of magnetic pulsations may be understood in terms of the coupling process.

This paper concentrates upon the resonant coupling of fast and Alfvén waves, which occurs when the frequency of the fast mode matches one of the natural Alfvén frequencies. In a nonuniform medium the natural Alfvén frequencies vary from one field line to another; thus the set of field lines which experience resonant Alfvén wave excitation is discrete, and in a simple system would form a sheet.

Previous studies have employed a variety of simplifications; often the dependence on ignorable coordinates is factored out and a harmonic time dependence is assumed which leaves an ordinary differential equation with a singularity at the resonant sheet.[1-3] Other studies have relaxed the assumption of harmonic time dependence, but still consider simple magnetic geometries. These partial differential equation problems are generally solved numerically[4-9] although analytical methods are sometimes tractable.[10]

Modeling has also been extended from simple geometries to include more general field geometries. In an effort to model magnetospheric phenomena the three-dimensional dipole field has received much attention from numerical and analytical calculations.[11-13] More general (two dimensional) systems have also been considered analytically for a cold plasma[14-18] and also for warm plasmas.[15,19,20]

In this paper, we focus upon the resonant excitation of Alfvén waves in a cold plasma permeated by a curl-free background magnetic field. The magnetic field is quite arbitrary, save that it possess an ignorable coordinate (β) and have no component of the background field along that direction ($\mathbf{B}\cdot\hat{\beta}=0$). For example, a poloidal field will satisfy these requirements, and we would identify the toroidal or azimuthal coordinate with β.

The dependence upon the ignorable coordinate and time is factored out as $\exp[i(k_\beta\beta-\omega t)]$; The wave number in the $\hat{\beta}$ direction is k_β. The governing equations may be reduced to two coupled partial differential equations, in contrast to the single ordinary differential equation found in simpler systems. The natural Alfvén frequencies in such a two-dimensional system depend only upon the poloidal transverse coordinate α. Thus the resonant singularity in the partial differential problem occurs on a flux surface labeled by constant α. The existence of a singularity in the pair of coupled PDE's (partial differential equations) is not obvious from a cursory inspection of the equations. Indeed, Hansen and Goertz[21] (hereafter HG) claim that "field line resonances" and the associated singularities are not present in the coupled PDE's we study here—a point to which we return.

In simple systems (governed by an ordinary differential equation: ODE) the singularity may be treated by means of a Frobenius series at the resonant singularity.[2,3,22,23] Whilst there is a well-defined procedure for constructing the two independent Frobenius series in these problems, there is no standard recipe to follow for the treatment of singularities in the partial differential equation system. One approach is to assume *a priori* an ordering between the width of the resonant sheet and the amplitude of the wave fields. Such an approach draws upon experience gained from simpler problems in selecting an appropriate ordering. Nevertheless, this method can demonstrate that the resonance is a logarithmic singularity;[11,15] however, this corresponds to only the first term in the singular solution. There is a second (regular) solution which cannot be determined by this approach.

In order to match given boundary conditions to the solution in the vicinity of the resonant singularity it is necessary to know the two independent solutions to the governing equations. If only one solution is known we cannot satisfy general boundary conditions.

HG criticize the method employed in studies such as Ref. 11 or Ref. 15 claiming that their solutions are incorrect, owing to neglect of some terms in an expansion. HG employ an alternative expansion procedure and find nonsingular solutions to the coupled PDE problem, concluding that nonuniformity along the background field destroys the singular nature of the waves.

Recently it has been demonstrated how a generalized Frobenius series may be derived which will give all of the terms in both regular and singular solutions at the singularity in a system of partial differential equations.[24] The systematic method devised there was applied to a medium containing a uniform magnetic field and a cold plasma whose density varied in the (x,z) plane, the field being aligned with $\hat{\mathbf{z}}$. In this paper, we generalize the method further to describe resonances in more general background magnetic-field geometries.

Our conclusions are contrary to those of HG; we find that there is a logarithmic singularity in the solution even when the background medium varies along the equilibrium field. Our calculation, which makes no *a priori* assumptions, vindicates the findings of a singular solution in studies such as Ref. 11 and contradicts the claims made by HG. We have submitted a comment[25] on the work of HG in which we demonstrate errors in their analysis.

The paper is structured as follows. Section II describes the magnetic field model and presents the governing equations. Section III describes the Alfvén eigenmodes and eigenfrequencies and some of their properties such as orthogonality and completeness. The Generalized Frobenius Series is presented in Sec. IV, while Sec. V considers the continuation of logarithmic terms across the resonant layer and the associated absorption of energy. Section VI describes how our series solution may be used in conjunction with a numerical solution. Our results are discussed and summarized in Sec. VII which concludes the main text. Much of the lengthy algebra involved in our calculation is reserved for the appendices. Appendix A deduces the indices for the regular and singular series, while Appendix B uses these indices to generate recursion relations and calculates the first few terms in both series.

II. GOVERNING EQUATIONS

The coordinate system used throughout this paper is an orthogonal curvilinear one based upon the magnetic geometry. We define three spatial coordinates (α,β,γ) and let $\hat{\gamma}$ be parallel to the local background magnetic-field direction everywhere. The transverse coordinates (α,β) are constant on any background line of force and are similar to Euler potentials or Clebsch variables. The background magnetic field is assumed to be solenoidal and irrotational, requiring

$$Bh_\alpha h_\beta = f(\alpha,\beta), \tag{1}$$

$$Bh_\gamma = g(\gamma), \tag{2}$$

where f and g are arbitrary functions of their arguments and the scale factors h_i are equal to $\hat{\mathbf{i}}/\nabla i$, where $i=\alpha,\beta,\gamma$. A physical interpretation of the scale factors may be realized by noting that a real-space element $d\mathbf{r}$ is equal to $\hat{\alpha}h_\alpha\, d\alpha + \hat{\beta}h_\beta d\beta + \hat{\gamma}h_\gamma\, d\gamma$. These results are standard properties of such a coordinate system.[26] Similar coordinate systems have facilitated earlier investigations of related problems.[27–31]

In the cold plasma limit the entire wave field can be described in terms of the transverse plasma displacements ξ_α and ξ_β. Factoring out a dependence of $\exp[i(k_\beta\beta-\omega t)]$ from all perturbations, the linearized momentum and time-integrated induction equations may be combined to give the following inhomogeneous wave equations:

$$\frac{\partial}{\partial\gamma}\left(\frac{h_\alpha}{h_\beta h_\gamma}\cdot\frac{\partial}{\partial\gamma}(\xi_\alpha h_\beta B)\right)+\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma}{h_\alpha h_\beta}\cdot\frac{\partial}{\partial\alpha}(\xi_\alpha h_\beta B)\right)$$
$$+\omega^2 h_\alpha h_\gamma\frac{B}{V^2}\xi_\alpha = -ik_\beta\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma B}{h_\beta}\xi_\beta\right) \tag{3a}$$

and

$$\frac{\partial}{\partial\gamma}\left(\frac{h_\beta}{h_\alpha h_\gamma}\cdot\frac{\partial}{\partial\gamma}(\xi_\beta h_\alpha B)\right)+\omega^2 h_\beta h_\gamma\frac{B}{V^2}\xi_\beta$$
$$=-ik_\beta\frac{h_\gamma}{h_\alpha h_\beta}\cdot\frac{\partial}{\partial\alpha}(\xi_\alpha h_\beta B)+k_\beta^2\frac{h_\gamma B}{h_\beta}\xi_\beta, \tag{3b}$$

where V is the Alfvén speed $[V^2=B^2/(\mu_0\rho_0)$; ρ_0 is the background plasma density]. Evidently, if $k_\beta=0$ the fast and Alfvén modes decouple: the fast mode being described by the plasma motion ξ_α confined to planes of constant β, while ξ_β represents axisymmetric toroidal Alfvén waves. Indeed, the left-hand side (LHS) of (3a) is the fast mode operator which depends upon derivatives both along and across the background magnetic field, and the LHS of (3b) is the Alfvén wave operator depending only upon field aligned derivatives.

It is evident from the form of (3) that in our coordinate system the quantities $\xi_\alpha h_\beta B$ and $\xi_\beta h_\alpha B$ are more convenient to work with than the plasma displacements ξ_α and ξ_β. [Given the general property of B in (1), it follows that $\xi_\alpha h_\beta B$ and $\xi_\beta h_\alpha B$ are proportional to the contravariant components of ξ_α and ξ_β, respectively.] We define the new variables $\tilde{\xi}_\alpha$, $\tilde{\xi}_\beta$, and $\tilde{b}_\gamma$ as

$$\tilde{\xi}_\alpha=\xi_\alpha h_\beta B;\quad \tilde{\xi}_\beta=\xi_\beta h_\alpha B;\quad \tilde{b}_\gamma=b_\gamma h_\alpha h_\beta, \tag{4}$$

where b_γ is the $\hat{\gamma}$ component of the magnetic field perturbation, and will be calculated later in this paper. We noted above that the LHS of (3b) was the Alfvén wave operator for Alfvén waves polarized in the $\hat{\beta}$ direction. The first and third terms on the LHS of (3a) may be thought of as the Alfvén wave operator for waves polarized in the $\hat{\alpha}$ direction (i.e., ξ_α), and it is useful to isolate these terms in the succeeding calculation. Denoting the Alfvén wave operators for the variables $\tilde{\xi}_\alpha$ and $\tilde{\xi}_\beta$ as D_α and D_β, respectively, (3) becomes

$$D_\alpha(\tilde{\xi}_\alpha)+\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma}{h_\alpha h_\beta}\cdot\frac{\partial\tilde{\xi}_\alpha}{\partial\alpha}\right)=-ik_\beta\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma}{h_\alpha h_\beta}\tilde{\xi}_\beta\right), \tag{5a}$$

$$D_\beta(\tilde{\xi}_\beta)=-ik_\beta\frac{h_\gamma}{h_\alpha h_\beta}\cdot\frac{\partial\tilde{\xi}_\alpha}{\partial\alpha}+k_\beta^2\frac{h_\gamma}{h_\alpha h_\beta}\tilde{\xi}_\beta, \tag{5b}$$

where

$$D_i(\tilde{\xi}_i)\equiv\frac{\partial}{\partial\gamma}\left(H^i_{j\gamma}\frac{\partial\tilde{\xi}_i}{\partial\gamma}\right)+\omega^2G^{i\gamma}_j\tilde{\xi}_i \tag{6a}$$

and

$$H^i_{j\gamma}=\frac{h_i}{h_jh_\gamma};\quad G^{i\gamma}_j=\frac{h_ih_\gamma}{h_jV^2}. \tag{6b}$$

The indices ij take the combinations $\alpha\beta$ and $\beta\alpha$.

III. ALFVÉN EIGENMODES AND EIGENFREQUENCIES

The locations of any resonances in the system of equations (5) are determined by the natural Alfvén frequencies of the operator D_β. If we assume suitable boundary conditions at the end of the field lines (e.g., the field passes through a perfectly conducting massive boundary on which both ξ_α and ξ_β would be zero) the equation $D_\beta(\tilde{\xi}_\beta)=0$ is a Sturm–Liouville problem. The same result is also true if the field lines are closed, as in some laboratory fusion devices. On any given field line (i.e., any α) there will be a discrete set of real natural frequencies $\{\omega_n\}$ corresponding to the oscillation frequencies of the eigenmodes $\tilde{\xi}_{\beta n}$. The eigenmodes are similar to the displacement eigenmodes of waves on a nonuniform string (cf. Ref. 32).

The nth eigenmode $\tilde{\xi}_{\beta n}$ and eigenfrequency ω_n satisfy the following equation (subject to $\tilde{\xi}_{\beta n}=0$ at the boundaries):

$$\frac{\partial}{\partial\gamma}\left(H^\beta_{\alpha\gamma}\frac{\partial\tilde{\xi}_{\beta n}}{\partial\gamma}\right)+\omega_n^2G^{\beta\gamma}_\alpha\tilde{\xi}_{\beta n}=0. \tag{7}$$

This equation may be solved at every value of α to produce the series of eigenfunctions and eigenfrequencies $\tilde{\xi}_{\beta n}(\alpha,\gamma)$ and $\omega_n(\alpha)$, where n may take the values $1,2,3,\ldots(n=1$ corresponding to the fundamental mode). In the functions $\tilde{\xi}_{\beta n}(\alpha,\gamma)$ the coordinate α is a parameter defining the field line of interest for which Eq. (7) is then solved as a function of γ. Combining (6) and (7) we find

$$D_\beta(\tilde{\xi}_{\beta n})=(\omega^2-\omega_n^2)G^{\beta\gamma}_\alpha\tilde{\xi}_{\beta n} \tag{8}$$

which will be of use in the following sections.

The eigenmodes form a complete orthogonal set on each field line. The orthogonality property between any two modes (on the same field line) may be expressed in the form

$$\int_{\gamma_1(\alpha)}^{\gamma_2(\alpha)}\tilde{\xi}_{\beta n}\tilde{\xi}_{\beta m}G^{\beta\gamma}_\alpha d\gamma=0,\quad n\neq m. \tag{9}$$

The integral is performed along a field line (α=const) between the boundaries $\gamma_1(\alpha)$ and $\gamma_2(\alpha)$.

The completeness of the modes on any given field line allows us to write an arbitrary function p (which satisfies the same boundary conditions as ξ_β) as a sum over the eigenmodes of that field line weighted with appropriate coefficients $\{p_n(\alpha)\}$,

$$p(\alpha,\gamma)=\sum_{n=1}^{\infty}p_n(\alpha)\tilde{\xi}_{\beta n}(\alpha,\gamma). \tag{10}$$

Once again, α should be thought of as a parameter which defines the field line of interest. A restriction on our model should be noted here. In the succeeding sections we shall write the displacements on field lines near the resonant surface as a sum over the eigenfunctions of the resonant surface. This is only possible if the displacements share common boundary conditions in γ, requiring that any boundaries lie perpendicular to the background magnetic field.

A. Inversion of the Alfvén wave operator

In the following sections it is sometimes necessary to invert the Alfvén wave operator: i.e., if $D_\beta[p(\alpha,\gamma)]=q(\alpha,\gamma)$, and q is given, what is p? If we write p as a sum (10), then the problem is to determine the coefficients p_n, which obey

$$\sum_{n=1}^{\infty}(\omega^2-\omega_n^2)G^{\beta\gamma}_\alpha p_n\tilde{\xi}_{\beta n}=q. \tag{11}$$

The relation (8) was used in obtaining the above result.

We may isolate each coefficient in the sum in (11) by multiplying the whole equation through by $\tilde{\xi}_{\beta n}$ ($n=1,2,3,\ldots$) and integrating along the field line of interest. For these purposes it is convenient to define the operator

$$\langle A(\alpha,\gamma)\rangle=\int_{\gamma_1(\alpha)}^{\gamma_2(\alpha)}A\,d\gamma \tag{12}$$

and recalling (9) we find the result

$$(\omega^2-\omega_n^2)p_n\langle\tilde{\xi}^2_{\beta n}G^{\beta\gamma}_\alpha\rangle=\langle\tilde{\xi}_{\beta n}q\rangle. \tag{13}$$

The $\langle - \rangle$ terms of (13) can always be evaluated (this may need to be done numerically for complicated media), and thus we can determine all of the coefficients $\{p_n\}$ provided that ω does not match any of the natural eigenfrequencies ω_n ($n=1,2,3,\ldots$). If one of the Alfvén eigenmodes is resonant, then the inversion for p may only be done up to an arbitrary component of the resonant eigenfunction: specifically, if the $n=r$ mode is resonant (i.e., $\omega^2=\omega_r^2$), then the function p may be expressed in terms of q in the form of a nonresonant component $[p^{nr}(\alpha,\gamma)]$ and resonant component (proportional to $\tilde{\xi}_{\beta r}$)

$$p(\alpha,\gamma)=p^{nr}(\alpha,\gamma)+\epsilon\tilde{\xi}_{\beta r}(\alpha,\gamma), \tag{14a}$$

where the nonresonant component is defined as

$$p^{nr}(q,\alpha,\gamma)=\sum_{n\neq r}\frac{\langle\tilde{\xi}_{\beta n}q(\alpha,\gamma)\rangle}{(\omega^2-\omega_n^2)\langle\tilde{\xi}_{\beta n}^2(\alpha,\gamma)G_\alpha^{\beta\gamma}(\alpha,\gamma)\rangle}\tilde{\xi}_{\beta n}(\alpha,\gamma). \tag{14b}$$

The constant ϵ may not be determined without additional information. Moreover, it is evident from (13) that in the resonant case the function q is not completely arbitrary, but must satisfy the solvability condition

$$\langle\tilde{\xi}_{\beta r}q\rangle=0, \quad \text{if } \omega^2=\omega_r^2. \tag{15}$$

IV. GENERALIZED FROBENIUS SERIES

The coupled equations (5) contain a singularity at any value of α where one of the natural Alfvén eigenfrequencies matches the frequency ω. We shall consider the solution to the equations in the vicinity of one of these resonant field lines, and without loss of generality choose our coordinate α to have its origin at the resonance, i.e., $\omega_r(0)=\omega$ for the resonant rth harmonic.

If we consider the variation of the wave fields on crossing the singularity at constant γ, we would expect to be able to write the variation of the fields in α as a Frobenius series. (Assuming that the system possesses a regular, rather than irregular, singularity—as is the case for simpler models.) Now consider moving along the resonant surface in γ: It should be possible to repeat the argument and write the variation of the wave fields in α as a Frobenius series at the new value of γ. Of course, the coefficients in the two series will be different. If the medium is continuous, it is likely that we should be able to write the wave fields in the vicinity of the resonance as a single Frobenius series whose coefficients are continuous functions of γ,

$$\begin{pmatrix}\tilde{\xi}_\alpha\\ \tilde{\xi}_\beta\end{pmatrix}=\sum_{n=0}^{\infty}\begin{pmatrix}a_n(\gamma)\\ b_n(\gamma)\end{pmatrix}\alpha^{\sigma+n}+\ln(\alpha)\sum_{n=0}^{\infty}\begin{pmatrix}c_n(\gamma)\\ d_n(\gamma)\end{pmatrix}\alpha^{\sigma+n}, \tag{16}$$

where not all of a_0, b_0, c_0 and d_0 are zero for definiteness, to fix the permitted values of σ. Note that these coefficients are solely a function of γ. It is convenient to let α be a dimensionless coordinate (e.g., the L-shell parameter). In this case h_α will have dimensions of length, and $h_\alpha\alpha$ will represent the distance from the resonant surface for small values of α.

The series expansion in α requires that we expand the functions $H_{i\gamma}^j$ and $G_i^{j\gamma}$ as Taylor series in α;

$$H_{j\gamma}^i(\alpha,\gamma)=H_{j\gamma 0}^i(0,\gamma)+\alpha H_{j\gamma 1}^i(0,\gamma)+\alpha^2H_{j\gamma 2}^i(0,\gamma)+\cdots, \tag{17a}$$

$$G_j^{i\gamma}(\alpha,\gamma)=G_{j0}^{i\gamma}(0,\gamma)+\alpha G_{j1}^{i\gamma}(0,\gamma)+\alpha^2G_{j2}^{i\gamma}(0,\gamma)+\cdots, \tag{17b}$$

where

$$H_{j\gamma n}^i(0,\gamma)=\frac{1}{n!}\cdot\left.\frac{\partial^nH_{j\gamma}^i}{\partial\alpha^n}\right|_{\alpha=0}; \qquad G_{jn}^{i\gamma}(0,\gamma)=\frac{1}{n!}\cdot\left.\frac{\partial^nG_j^{i\gamma}}{\partial\alpha^n}\right|_{\alpha=0}. \tag{18}$$

Thus expanded wave operators D_i take the form

$$D_i(\tilde{\xi}_i)=D_{i0}(\tilde{\xi}_i)+\alpha D_{i1}(\tilde{\xi}_i)+\alpha^2D_{i2}(\tilde{\xi}_i)+\cdots \tag{19a}$$

and

$$D_{in}(\tilde{\xi}_i)=\frac{\partial}{\partial\gamma}\left(H_{j\gamma n}^i\frac{\partial\tilde{\xi}_i}{\partial\gamma}\right)+\omega^2G_{jn}^{i\gamma}\tilde{\xi}_i. \tag{19b}$$

The governing coupled equations (5) may be written in the expanded form

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}\alpha^nD_{\alpha n}\right)\cdot\left(\sum_{n=0}^{\infty}a_n\alpha^{\sigma+n}+\ln(\alpha)\sum_{n=0}^{\infty}c_n\alpha^{\sigma+n}\right)+\left(\sum_{n=0}^{\infty}n\alpha^{n-1}H_{\alpha\beta n}^\gamma\right)\cdot\left(\sum_{n=0}^{\infty}[a_n(\sigma+n)+c_n]\alpha^{\sigma+n-1}\right.\\&\left.+\ln(\alpha)\sum_{n=0}^{\infty}c_n(\sigma+n)\alpha^{\sigma+n-1}\right)+\left(\sum_{n=0}^{\infty}\alpha^nH_{\alpha\beta n}^\gamma\right)\cdot\left(\sum_{n=0}^{\infty}[a_n(\sigma+n)(\sigma+n-1)\right.\\&\left.+c_n(2\sigma+2n-1)]\alpha^{\sigma+n-2}+\ln(\alpha)\sum_{n=0}^{\infty}c_n(\sigma+n)(\sigma+n-1)\alpha^{\sigma+n-2}\right)\\&=-ik_\beta\left(\sum_{n=0}^{\infty}n\alpha^{n-1}H_{\alpha\beta n}^\gamma\right)\cdot\left(\sum_{n=0}^{\infty}b_n\alpha^{\sigma+n}+\ln(\alpha)\sum_{n=0}^{\infty}d_n\alpha^{\sigma+n}\right)\\&\quad-ik_\beta\left(\sum_{n=0}^{\infty}\alpha^nH_{\alpha\beta n}^\gamma\right)\cdot\left(\sum_{n=0}^{\infty}[b_n(\sigma+n)+d_n]\alpha^{\sigma+n-1}+\ln(\alpha)\sum_{n=0}^{\infty}d_n(\sigma+n)\alpha^{\sigma+n-1}\right)\end{aligned} \tag{20a}$$

and

$$\left(\sum_{n=0}^{\infty}\alpha^nD_{\beta n}\right)\cdot\left(\sum_{n=0}^{\infty}b_n\alpha^{\sigma+n}+\ln(\alpha)\sum_{n=0}^{\infty}d_n\alpha^{\sigma+n}\right)$$
$$=-ik_\beta\left(\sum_{n=0}^{\infty}\alpha^nH^{\gamma}_{\alpha\beta n}\right)\cdot\left(\sum_{n=0}^{\infty}[a_n(\sigma+n)+c_n]\alpha^{\sigma+n-1}+\ln(\alpha)\sum_{n=0}^{\infty}c_n(\sigma+n)\alpha^{\sigma+n-1}\right)$$
$$+k_\beta^2\left(\sum_{n=0}^{\infty}\alpha^nH^{\gamma}_{\alpha\beta n}\right)\cdot\left(\sum_{n=0}^{\infty}b_n\alpha^{\sigma+n}+\ln(\alpha)\sum_{n=0}^{\infty}d_n\alpha^{\sigma+n}\right), \tag{20b}$$

where we have performed the derivatives with respect to α required by (5).

The majority of the remainder of this paper demonstrates how to determine, in a systematic fashion, the sets of Frobenius coefficients in (16) which satisfy the governing expansions (20).

In simple systems (where we are only required to solve an ordinary differential equation) the lowest order terms of (20) would yield an indicial equation specifying two roots of σ for which there are nonzero solutions. The first step is not so simple in our more general calculation; however, in Appendix A we prove that if σ has any value other than 0 or -1 we require $a_0=b_0=c_0=d_0=0$, which for definiteness is not allowed. The permitted values of the index σ are the same as in simpler calculations, and in the following sections we show that $\sigma=-1(0)$ corresponds to the singular (regular) solution.

To construct the regular and singular solutions we set the index, σ, equal to 0 and -1, respectively, and solve the relations resulting from equating the coefficients of each order of α^n and $\alpha^n\ln(\alpha)$ in (20). The process is rather lengthy, but straightforward. The details are given in Appendix B, but here in the main text we shall only quote the results. The appendices yield the displacements $\tilde{\xi}_\alpha$ and $\tilde{\xi}_\beta$. It is also useful to calculate the magnetic field pressure perturbation, and so we also quote the compressional magnetic-field perturbation b_{γ} expressed in the form of $\tilde{b}_\gamma$ [see Eq. (4)]. The quantity $\tilde{b}_\gamma$ is calculated from the $\hat{\gamma}$ component of the induction equation, which takes the form

$$\tilde{b}_\gamma=-\frac{\partial\tilde{\xi}_\alpha}{\partial\alpha}-ik_\beta\tilde{\xi}_\beta. \tag{21}$$

In order to reduce the length of many of the expressions obtained in the appendices, we shall define the following notation: The nonresonant components of the coefficients b and d are solely a function of γ—assuming that q is a known function [see (14)]. In the appendices we need only consider the quantities on the resonant surface at $\alpha=0$, and we define [e.g., (26a) and (26b) below]

$$b^{nr}\{q_0\}\quad\text{or}\quad d^{nr}\{q_0\}$$
$$=\sum_{n\neq r}\frac{\langle\tilde{\xi}_{\beta n}q_0\rangle}{(\omega^2-\omega_n^2)\langle\tilde{\xi}^2_{\beta n}(0,\gamma)G^{\beta\gamma}_{\alpha}(0,\gamma)\rangle}\tilde{\xi}_{\beta n}(0,\gamma). \tag{22a}$$

The function $q_0(\gamma)=q(0,\gamma)$ when comparing with (14b), and all quantities and integrations are evaluated on the surface $\alpha=0$. Another cumbersome expression is produced when applying the solvability condition (15) to determine quantities such as β and δ; these constants represent the coefficients of the resonant eigenfunction in expansions of b and d—cf. ϵ in (14a). We shall define

$$\beta\{q_0\}\quad\text{or}\quad\delta\{q_0\}=\frac{\langle\tilde{\xi}_{\beta r}q_0\rangle}{\langle\tilde{\xi}_{\beta r}D_{\beta 1}\tilde{\xi}_{\beta r}\rangle} \tag{22b}$$

and once again, everything is evaluated on the $\alpha=0$ surface.

A. The regular Generalized Frobenius Series

Adopting the $\sigma=0$ index, we may generate the regular solution which has the form

$$\tilde{\xi}_\alpha(\alpha,\gamma)=a_0(\gamma)+a_1(\gamma)\alpha+a_2(\gamma)\alpha^2+O(\alpha^3), \tag{23a}$$
$$\tilde{\xi}_\beta(\alpha,\gamma)=b_0(\gamma)+b_1(\gamma)\alpha+b_2(\gamma)\alpha^2+O(\alpha^3), \tag{23b}$$
$$\tilde{b}_\gamma(\alpha,\gamma)=-[ik_\beta b_0(\gamma)+a_1(\gamma)]-[ik_\beta b_1(\gamma)+2a_2(\gamma)]\alpha+O(\alpha^2). \tag{23c}$$

Apart from satisfying the boundary contitions at the ends of the field lines, the only other constraint upon the coefficients is a requirement of a_0 and b_0 (B8b),

$$\langle\tilde{\xi}_{\beta r}[D_{\beta 1}(b_0)-ik_\beta D_{\alpha 0}(a_0)]\rangle=0. \tag{24a}$$

The other coefficients may be determined in terms of a_0 and b_0:

$$a_1=(i/k_\beta H^{\gamma}_{\alpha\beta 0})D_{\beta 0}(b_0)-ik_\beta b_0, \tag{24b}$$
$$b_1=b_1^{nr}+\beta_1\tilde{\xi}_{\beta r}, \tag{24c}$$
$$b_1^{nr}=b_1^{nr}\{ik_\beta D_{\alpha 0}a_1-D_{\beta 1}b_0\}, \tag{24d}$$
$$\beta_1=\beta_1\{(ik_\beta/2)[D_{\alpha 0}(a_1)+D_{\alpha 1}(a_0)]-D_{\beta 1}(b_1^{nr})-D_{\beta 2}(b_0)\}, \tag{24e}$$
$$a_2=(1/2H^{\gamma}_{\alpha\beta 0})[-ik_\beta(H^{\gamma}_{\alpha\beta 1}b_0+H^{\gamma}_{\alpha\beta 0}b_1)-D_{\alpha 0}(a_0)-H^{\gamma}_{\alpha\beta 1}a_1], \tag{24f}$$
$$b_2=b_2^{nr}+\beta_2\tilde{\xi}_{\beta r}, \tag{24g}$$
$$b_2^{nr}=b_2^{nr}\{(ik_\beta/2)[D_{\alpha 0}(a_1)+D_{\alpha 1}(a_0)]-D_{\beta 1}(b_1)-D_{\beta 2}(b_0)\}, \tag{24h}$$
$$\beta_2=\beta_2\{(ik_\beta/3)[D_{\alpha 0}(a_2)+D_{\alpha 1}(a_1)+D_{\alpha 2}(a_0)]-D_{\beta 1}(b_2^{nr})-D_{\beta 2}(b_1)-D_{\beta 3}(b_0)\}. \tag{24i}$$

B. The singular Generalized Frobenius Series

Adopting the $\sigma=-1$ index the singular solution may be generated, and has the form

$$\tilde{\xi}_\alpha(\alpha,\gamma)=-ik_\beta\beta_0\tilde{\xi}_{\beta r}(0,\gamma)\ln(\alpha)+a_2(\gamma)\alpha -ik_\beta\delta_1\tilde{\xi}_{\beta r}(0,\gamma)\alpha\ln(\alpha) +O[\alpha^2,\alpha^2\ln(\alpha)], \tag{25a}$$

$$\tilde{\xi}_\beta(\alpha,\gamma)=\frac{\beta_0\tilde{\xi}_{\beta r}(0,\gamma)}{\alpha}+\beta_1\tilde{\xi}_{\beta r}(0,\gamma)+\delta_1\tilde{\xi}_{\beta r}(0,\gamma)\ln(\alpha) +b_2(\gamma)\alpha+d_2(\gamma)\alpha\ln(\alpha) +O[\alpha^2,\alpha^2\ln(\alpha)], \tag{25b}$$

$$\tilde{b}_\gamma(\alpha,\gamma)=-i\frac{\beta_0}{k_\beta H^\gamma_{\alpha\beta0}}D_{\beta1}\tilde{\xi}_{\beta r}(0,\gamma)+O[\alpha,\alpha\ln(\alpha)], \tag{25c}$$

where β_0 is the amplitude of the singular solution, and the other parameters above are defined as

$$\beta_1=\beta_1\{-\beta_0D_{\beta2}\tilde{\xi}_{\beta r}-\delta_1D_{\beta1}\tilde{\xi}_{\beta r}\}, \tag{26a}$$

$$\delta_1=\delta_1\{k_\beta^2\beta_0D_{\alpha0}\tilde{\xi}_{\beta r}\}, \tag{26b}$$

$$a_2=i\frac{\beta_0}{k_\beta H^\gamma_{\alpha\beta0}}D_{\beta1}\tilde{\xi}_{\beta r}(0,\gamma)+ik_\beta(\delta_1-\beta_1)\tilde{\xi}_{\beta r}(0,\gamma), \tag{26c}$$

$$b_2=b_2^{nr}+\beta_2\tilde{\xi}_{\beta r}(0,\gamma), \tag{26d}$$

$$b_2^{nr}=b_2^{nr}\{-(\beta_1+\delta_1)D_{\beta1}\tilde{\xi}_{\beta r}-\beta_0D_{\beta2}\tilde{\xi}_{\beta r}-D_{\beta0}d_2^{nr}\}, \tag{26e}$$

$$\beta_2=\beta_2\{\tfrac{1}{2}(ik_\beta D_{\alpha0}a_2-D_{\beta1}d_2^{nr}-\delta_2D_{\beta1}\tilde{\xi}_{\beta r}-\delta_1D_{\beta2}\tilde{\xi}_{\beta r}) -D_{\beta1}b_2^{nr}-\beta_1D_{\beta2}\tilde{\xi}_{\beta r}-\beta_0D_{\beta3}\tilde{\xi}_{\beta r}\}, \tag{26f}$$

$$d_2=d_2^{nr}+\delta_2\tilde{\xi}_{\beta r}(0,\gamma), \tag{26g}$$

$$d_2^{nr}=d_2^{nr}\{k_\beta^2\beta_0D_{\alpha0}\tilde{\xi}_{\beta r}-\delta_1D_{\beta1}\tilde{\xi}_{\beta r}\}, \tag{26h}$$

$$\delta_2=\delta_2\{(k_\beta^2/2)(\delta_1D_{\alpha0}\tilde{\xi}_{\beta r}+\beta_0D_{\alpha1}\tilde{\xi}_{\beta r})-D_{\beta1}d_2^{nr} -\delta_1D_{\beta2}\tilde{\xi}_{\beta r}\}. \tag{26i}$$

The procedure for determining the coefficients of both the regular and singular series to all orders is given in Appendix B. The existence of the logarithmic terms in the singular solution is in accord with the results of earlier studies[2,3,11,15,16,24,33] except that of HG who disagree with all of these studies and claim that there is no singularity. Taking the appropriate limit of our solution recovers the results of previous exact calculations in the ODE problem[34] and the PDE problem[22]—see Ref. 24 for a discussion. In a separate communication[25] we examine the analysis of HG, and find that it contains errors.

V. CONTINUING THE SOLUTION ACROSS THE SINGULARITY

To complete the singular solution it is necessary to determine how to continue the solution across the singularity. One way to do this is to allow the boundaries at the ends of the field lines to be weakly absorptive.[15,34] In our approach, which is based upon that of Ref. 15, we consider that the boundaries are not absorptive but rather that the driving of the system increases slowly, so that the time dependence of the driving is $\exp(-i\omega t)$, where now $\omega=\omega_R+i\omega_I$ (ω_R and ω_I are real, $\omega_I>0$). The driver amplitude grows over a time of order $2\pi/\omega_I$ which is assumed to be much greater than any Alfvén or fast mode transit time of the system—thus all wave fields may be assumed to have a time dependence of $\exp(-i\omega t)$. Here ω_R is the real driving frequency that was written in previous sections without a subscript. Thus ω_R^2 is equal to the square of the rth eigenfrequency of the $\alpha=0$ field line, viz. $\omega_R^2=\omega_r^2(0)$.

When ω is complex there is no longer an exact resonance at $\alpha=0$ (or any other real α). In a formal sense, the resonance now lies instead at a complex position $\alpha_{\rm res}=0+i\alpha_i$. Of course, α_i is a function of ω_I. (Note that at this stage α_i may be complex.) We seek to determine the dependence of α_i upon ω_I, most importantly its sign.

In this section we shall write

$$\frac{\partial}{\partial\gamma}\left(H^\beta_{\alpha\gamma}\frac{\partial\tilde{\xi}_\beta(\alpha,\gamma)}{\partial\gamma}\right)+\omega^2G^{\beta\gamma}_\alpha\tilde{\xi}_\beta(\alpha,\gamma) \equiv D_\beta(\omega,\alpha)\tilde{\xi}_\beta(\alpha,\gamma). \tag{27}$$

Here, $D_\beta(\omega,\alpha)$ has an implicit γ dependence, which should be understood. However, we must also allow for the possibility of evaluating D_β at complex positions α and for complex frequencies ω, both of which are stated explicitly for the remainder of this section. (Elsewhere in the paper, we have only had to consider real α and ω.) As before, the operator D_β can be expanded in the Taylor expansion about the resonant position $\alpha_{\rm res}$, which is now complex:

$$D_\beta(\omega,\alpha)=D_\beta(\omega,\alpha_{\rm res})+(\alpha-\alpha_{\rm res})D_{\beta1}(\omega,\alpha_{\rm res})+\cdots. \tag{28}$$

Now $D_\beta(\omega,\alpha_{\rm res})$ can in turn be expressed as a Taylor expansion about the original resonance, at $\alpha=0$:

$$D_\beta(\omega,\alpha_{\rm res})=D_\beta(\omega_r,0)+i\alpha_iD_{\beta1}(\omega_r,0) +i\omega_I\left.\frac{\partial D_\beta}{\partial\omega}\right|_{(\omega_r,0)}+\cdots. \tag{29}$$

The choice of driving frequency determines both the position $\alpha_{\rm res}$ of the resonance and the form of the resonant eigenfunction $\tilde{\xi}_{\beta\,\rm res}$, where

$$D_\beta(\omega,\alpha_{\rm res})\tilde{\xi}_{\beta\,\rm res}=0. \tag{30}$$

We can write $\tilde{\xi}_{\beta\,\rm res}$ in terms of the eigenfunction $\tilde{\xi}_{\beta r}$ at $\alpha=0$ with eigenfrequency ω_r as

$$\tilde{\xi}_{\beta\,\rm res}=\tilde{\xi}_{\beta r}+\Delta\tilde{\xi}_{\beta\,\rm res}. \tag{31}$$

Using expression (29) up to first order in small quantities for $D_\beta(\omega,\alpha_{\rm res})$ and expression (31) for $\tilde{\xi}_{\beta\,\rm res}$, Eq. (30) yields

$$i\alpha_i D_{\beta 1}(\omega_r,0)\tilde{\xi}_{\beta r}+i\omega_I \left.\frac{\partial D_\beta}{\partial\omega}\right|_{(\omega_r,0)}\tilde{\xi}_{\beta r}+D_\beta(\omega_r,0)\Delta\tilde{\xi}_{\beta\,\rm res}=0 \tag{32}$$

using $D_\beta(\omega_r,0)\tilde{\xi}_{\beta r}=0$. Multiplying by $\tilde{\xi}_{\beta r}$ and integrating with respect to γ at $\alpha=0$, we find

$$\alpha_i=-\omega_I\left\langle \tilde{\xi}_{\beta r}\left.\frac{\partial D_\beta}{\partial\omega}\right|_{(\omega_r,0)}\tilde{\xi}_{\beta r}\right\rangle \Big/ \langle\tilde{\xi}_{\beta r}D_{\beta 1}(\omega_r,0)\tilde{\xi}_{\beta r}\rangle, \tag{33}$$

where the operators $\partial D_\beta/\partial\omega$ and $D_{\beta 1}$ are evaluated at $\omega=\omega_r$ and $\alpha=0$. Note that the final step is possible because $\Delta\tilde{\xi}_{\beta\,\rm res}$ satisfies the same perfectly reflecting boundary conditions as $\tilde{\xi}_{\beta r}$ at the ends of the field lines. If the boundaries were weakly absorptive, there would be an additional term in Eqs. (32) and (33). For our differential operator the factor $\partial D_\beta/\partial\omega|_{(\omega_r,0)}=2\omega_r G^{\beta\gamma}_{\alpha 0}$. Note that this result implies that α_i is real to this order.

The Frobenius series solutions for the growing driver case are the same as before, except that instead of being expansions of powers of $(\alpha-0)$ and its logarithm, they are expansions in powers of $(\alpha-\alpha_{\rm res})\equiv(\alpha-i\alpha_i)$ and its logarithm. Consider then the argument of $\ln(\alpha-i\alpha_i)$ as one goes from $\mathrm{Re}[\alpha]<0$ to $\mathrm{Re}[\alpha]>0$. If α_i is positive, $\arg(\alpha-i\alpha_i)$ increases; whereas if α_i is negative it decreases. Finally, we let ω_I, $\alpha_i\to 0$: $\arg(\alpha-i\alpha_i)$ increases/decreases by π on moving from $\alpha=0^-$ to $\alpha=0^+$, and this determines how to continue the solution across the resonance in the case of a real driving frequency.

Having taken the limit $\omega_I\to 0$, the α component of the plasma displacement is found to be $\tilde{\xi}_\alpha\simeq -ik_\beta\beta_0\tilde{\xi}_{\beta r}\ln|\alpha|$ at $\alpha=0^-$ and $\tilde{\xi}_\alpha\simeq -ik_\beta\beta_0\tilde{\xi}_{\beta r}(\ln|\alpha|+i\pi\,\mathrm{sign}[\alpha_i])$ at $\alpha=0^+$. (Of course, we mean $\mathrm{sign}[\alpha_i]$, $\mathrm{sign}[\omega_I]$ to be evaluated before taking the limit.) This procedure may be repeated for the other logarithmic terms in the singular series. Thus given the solution on one side of the resonance, we can construct the corresponding singular series solution on the other side.

In the previous studies of ODE resonance problems,[2] it is clear that the change in phase of the argument of the logarithm depends solely upon the gradient of the rth Alfvén eigenfrequency at the resonance. The choice of sign of the phase change means that the resonance absorbs energy, rather than radiates energy. It is not evident from our more general analysis [see (33)] how α_i depends on $d\omega_r^2(\alpha)/d\alpha$. For example, is it possible to specify a D_β which will yield an arbitrary eigenfrequency gradient across the resonance? The factor $\langle\tilde{\xi}_{\beta r}D_{\beta 1}\tilde{\xi}_{\beta r}\rangle$ enters many of our expressions [e.g., when we apply the solvability condition (22b)], and it is worth developing an interpretation for it. Let the rth eigenmode and eigenfrequency satisfy $D_{\beta 0}\tilde{\xi}_{\beta r}=0$ at $\alpha=0$. Now consider the change in eigenmode and eigenfrequency on an adjacent field line at $\alpha=\delta\alpha$ ($\delta\alpha$ real). The change to the density, eigenfrequency, and eigenmode relative to those on $\alpha=0$ are $\delta\rho_0=\delta\alpha[\partial\rho_0(\alpha,\gamma)/\partial\alpha]_0$, $\delta\omega_r^2=\delta\alpha(d\omega_r^2/d\alpha)_0$ and $\delta\tilde{\xi}_{\beta r}=\delta\alpha[\partial\tilde{\xi}_{\beta r}(\alpha,\gamma)/\partial\alpha]_0$, respectively, all the derivatives being evaluated at $\alpha=0$. Substituting these changes in to a Taylor series expansion of $D_\beta\tilde{\xi}_{\beta r}=0$ about $\alpha=0$ we find the terms linear in $\delta\alpha$ satisfy

$$D_{\beta 0}\left.\frac{\partial\tilde{\xi}_{\beta r}}{\partial\alpha}\right|_{\alpha=0}+\left(D_{\beta 1}+G^{\beta\gamma}_{\alpha 0}\left.\frac{d\omega_r^2}{d\alpha}\right|_{\alpha=0}\right)\tilde{\xi}_{\beta r}(0,\gamma)=0. \tag{34}$$

Multiplying Eq. (34) by $\tilde{\xi}_{\beta r}$ and integrating along the background field line $\alpha=0$ we find the simple relation

$$\left.\frac{d\omega_r^2}{d\alpha}\right|_{\alpha=0}=-\frac{\langle\tilde{\xi}_{\beta r}D_{\beta 1}\tilde{\xi}_{\beta r}\rangle}{\langle\tilde{\xi}_{\beta r}G^{\beta\gamma}_{\alpha 0}\tilde{\xi}_{\beta r}\rangle}. \tag{35}$$

Recalling that $\partial D_{\beta 0}/\partial\omega^2|_{(\omega_r,0)}=G^{\beta\gamma}_{\alpha 0}$ it is clear from (33) and (35) that

$$\alpha_i=2\omega_r\omega_I\Big/\left(\frac{d\omega_r^2(\alpha)}{d\alpha}\right)_{\alpha=0}\equiv\omega_I\Big/\left(\frac{d\omega_r(\alpha)}{d\alpha}\right)_{\alpha=0}. \tag{36}$$

Thus, as one would anticipate, the sense of the phase change of the log terms across the resonance is determined by the gradient of the resonant eigenfrequency. Note that the signs of α_i, ω_I, and $d\omega_r/d\alpha$ are related by $\mathrm{sign}[d\omega_r/d\alpha]=\mathrm{sign}[\alpha_i]$, since ω_I is by assumption positive.

The resonant layer actually absorbs energy and is a consequence of the jump in phase of ξ_α, which leads to a discontinuous Poynting flux across the resonant layer. The α component of the Poynting flux, S_α, which will not time average to zero is $\mu_0^{-1}B\,\mathrm{Re}[b_\gamma\exp i(k_\beta\beta-\omega t)]\times\mathrm{Re}[v_\alpha\exp i(k_\beta\beta-\omega t)]$, where v_α is the plasma velocity in the α direction. After averaging with respect to time this is equivalent to $iB\omega(\xi_\alpha^* b_\gamma-\xi_\alpha b_\gamma^*)/(4\mu_0)$, where the perturbations in this expression have no implicit time dependence—i.e., they are related to the functions given in (23) and (25) via the definitions (4). The difference between the Poynting flux at $\alpha=0^-$ and that at 0^+ is the density of power absorbed by the resonant surface,

$$S_\alpha(0^-)-S_\alpha(0^+)$$
$$=-\mathrm{sign}[\alpha_i]\frac{\pi\omega|\beta_0|^2}{2\mu_0 h_\beta h_\gamma}\tilde{\xi}_{\beta r}(0,\gamma)\,D_{\beta 1}\tilde{\xi}_{\beta r}(0,\gamma), \tag{37}$$

where we have employed the definitions in (4) and (25). [Note that the regular solution makes no contribution to (37), since the regular solution is continuous at $\alpha=0$.]

An elemental area of the resonant surface is $h_\beta h_\gamma\,d\beta\,d\gamma$. Since our equilibrium is independent of β we may integrate the Poynting flux to find the power absorbed per unit of the β coordinate. Employing (35) we find

$$\langle[S_\alpha(0^-)-S_\alpha(0^+)]h_\beta h_\gamma\rangle$$
$$=\frac{\pi\omega^2}{\mu_0}|\beta_0|^2\langle\tilde{\xi}_{\beta r}G^{\beta\gamma}_{\alpha 0}\tilde{\xi}_{\beta r}\rangle\left|\frac{d\omega_r}{d\alpha}\right|_{\alpha=0}. \tag{38}$$

The modulus sign allows for the fact that $\text{sign}[d\omega_r/d\alpha]=\text{sign}[\alpha_i]$. The energy absorbed by the resonance is positive definite, as one would expect.

VI. NUMERICAL SOLUTIONS

One of the novel features of our analysis is the ability to derive all of the coefficients in the singular *and* regular series in a systematic fashion. Whilst numerical studies are suitable away from the resonance, it is impossible for them to resolve the singularity. A compromise is to integrate the governing equations numerically up to the neighborhood of the resonance, and then match onto an analytical series solution. This approach was adopted by Ref. 23 in the ODE problem and enabled them to calculate the quasi-eigenmodes of the system which oscillate and damp as energy is absorbed by the resonance. Solely numerical solutions, such as a finite element analysis will produce normal modes (with real eigenfrequency) and no net absorption at the resonance.[35] (Damped modes can be obtained by this method if the ionosphere is not a perfect reflector of energy.[36])

Let us suppose that the equations have been solved numerically up to some small value of α where the numerical scheme is still accurate. Such a solution can furnish us with the values of $\tilde{\xi}_\beta$, $\tilde{\xi}_\alpha$, $\partial\tilde{\xi}_\beta/\partial\alpha$, and $\partial\tilde{\xi}_\alpha/\partial\alpha$ along a field line at α, close to the resonance. Our series solution is determined by three factors; β_0, $a_0^r(\gamma)$ and $b_0^r(\gamma)$. The superscripts "r" and "s" will be used to avoid ambiguity between the coefficients in the regular and singular series. The first terms in our series solution are, from (23) and (25),

$$\tilde{\xi}_\alpha(\alpha,\gamma)=-ik_\beta\beta_0\tilde{\xi}_{\beta r}(0,\gamma)\ln(\alpha)+a_0^r(\gamma)+O[\alpha,\alpha\ln(\alpha)], \tag{39a}$$

$$\tilde{\xi}_\beta(\alpha,\gamma)=\frac{\beta_0}{\alpha}\tilde{\xi}_{\beta r}(0,\gamma)+\delta_1^s\ln(\alpha)\tilde{\xi}_{\beta r}(0,\gamma)+\beta_1^s\tilde{\xi}_{\beta r}(0,\gamma)+b_0^r(\gamma)+O[\alpha,\alpha\ln(\alpha)], \tag{39b}$$

$$\frac{\partial\tilde{\xi}_\alpha}{\partial\alpha}=-i\frac{\beta_0k_\beta}{\alpha}\tilde{\xi}_{\beta r}(0,\gamma)+O[1,\ln(\alpha)], \tag{39c}$$

$$\frac{\partial\tilde{\xi}_\beta}{\partial\alpha}=\frac{-\beta_0}{\alpha^2}\tilde{\xi}_{\beta r}(0,\gamma)+\frac{\delta_1^s}{\alpha}\tilde{\xi}_{\beta r}(0,\gamma)+O[1,\ln(\alpha)]. \tag{39d}$$

We only require three equations to determine β_0, $a_0^r(\gamma)$ and $b_0^r(\gamma)$. Since the omitted terms in (39c) are relatively more important than those omitted in (39d), we shall not use (39c). Taking the product of $\tilde{\xi}_{\beta r}(0,\gamma)G_{\alpha 0}^{\beta\gamma}$ and Eq. (39d) then integrating with respect to γ along the resonant field line we find

$$\beta_0=\left\langle\tilde{\xi}_{\beta r}G_{\alpha 0}^{\beta\gamma}\frac{\partial\tilde{\xi}_\beta}{\partial\alpha}\right\rangle\Bigg/\left[\langle\tilde{\xi}_{\beta r}G_{\alpha 0}^{\beta\gamma}\tilde{\xi}_{\beta r}\rangle\left(\frac{\delta_1^s}{\beta_0\alpha}-\frac{1}{\alpha^2}\right)\right],\quad \text{Lim }\alpha\to 0. \tag{40}$$

The term $\partial\tilde{\xi}_\beta/\partial\alpha$ is evaluated at small (but nonzero α) from a numerical integration. As the numerical solution is found at smaller and smaller α, we expect that $\tilde{\xi}_\beta$ will be proportional to α^{-1}, thus the numerator in (40) will behave like α^{-2}. When (40) is evaluated for $\alpha\to 0$ the value of the constant β_0 should be converged upon.

Once β_0 has been determined (39a) and (39b) may be used to calculate the functions $a_0^r(\gamma)$ and $b_0^r(\gamma)$, by comparing with the numerical solutions of $\tilde{\xi}_\alpha$ and $\tilde{\xi}_\beta$ at small, but finite, α. The solution may be continued across the singularity as described in Sec. V, and then used as initial conditions for the numerical solution on the other side of the resonance. Note that the regular coefficients must satisfy the constraint (24a), which may be used as a test of the accuracy of the numerical solution and the matching on to the Generalized Frobenius Series.

Although we have generally been concerned with real oscillation frequencies there is no reason why we should not consider complex frequencies.[23] In this case, the singularity occurs at a complex α coordinate, as we found in Sec. V. The branch cut from the singularity will cross the real α axis, which may be circumvented by deforming the contour into the complex α plane and around the singularity. In this case, the Generalized Frobenius Solution should be an expansion about a complex α.

VII. DISCUSSION AND SUMMARY

Many previous studies have shown how the amplitude of the singular solution (β_0) is sensitive to the overlap integral of the resonant eigenfunction and the compressional magnetic field along the resonant field line.[11,16,17,22] This quantity measures how effectively the magnetic pressure gradient ($\propto b_\gamma$) can drive the resonant eigenfunction. In nonuniform fields the geometry of the background will modify the efficiency of coupling, as we show below.

Note that the $\hat{\gamma}$ component of the time-integrated induction equation is $\tilde{b}_\gamma=-(\partial\tilde{\xi}_\alpha/\partial\alpha+ik_\beta\tilde{\xi}_\beta)$, and so (5b) may be written

$$D_\beta(\tilde{\xi}_\beta)=i\frac{k_\beta h_\gamma}{h_\alpha h_\beta}\tilde{b}_\gamma. \tag{41}$$

It is clear from the series solutions for b_γ in (23) and (25) that to lowest order $b_\gamma\sim O(\alpha^0)$, say, $b_{\gamma 0}(\gamma)$. Expanding the functions and operators in (41) in terms of α, multiplying by $\tilde{\xi}_{\beta r}$ and integrating along the resonant field line, we find to lowest order

$$\beta_0\langle\tilde{\xi}_{\beta r}D_{\beta 1}\tilde{\xi}_{\beta r}\rangle=ik_\beta\langle\tilde{\xi}_{\beta r}\tilde{b}_{\gamma 0}h_\gamma/(h_\alpha h_\beta)\rangle. \tag{42}$$

Employing (35), and recalling the definitions (4) and (6b) we may write β_0 in terms of the more physical quantities ξ_β and b_γ,

$$\beta_0=-i\frac{k_\beta}{2\mu_0\omega_r}\frac{\langle Bh_\alpha h_\gamma\xi_{\beta r}b_{\gamma 0}\rangle}{\langle\xi_{\beta r}\rho\xi_{\beta r}h_\alpha h_\beta h_\gamma\rangle}\left(\frac{d\omega_r}{d\alpha}\right)_0^{-1}. \tag{43}$$

This is consistent with the coupling efficiency found in Eq. (15) of Ref. 11, and Eq. (17) of Ref. 17. The integral in the denominator of (43) is simply a normalizing factor, while the integral in the numerator tells us how effectively the fast mode ($b_{\gamma 0}$) can drive the resonant eigenfunction ($\xi_{\beta r}$). It is interesting to convert this factor into an explicit integral along the length of the resonant field line. The angle brackets represent integration with respect to γ. An element along the field line of $d\gamma$ has a physical length of $dl = h_\gamma d\gamma$. Moreover, from Eq. (1) we know $Bh_\alpha = f(\alpha,\beta)/h_\beta$, and since we integrate at constant α and β, the efficiency of coupling is proportional to a factor

$$\int \frac{\xi_{\beta r} b_{\gamma 0}}{h_\beta} dl \tag{44}$$

which suggests greater efficiency when h_β is smallest—i.e., the separation of field lines in the invariant direction is small. For planetary magnetospheres this occurs at higher latitudes on the portions of the field line close to the ionosphere. Besides the geometrical effect of the background the factor (44) also depends upon the structure of the compressional wave field and the resonant eigenfunction. The form of these waves will depend upon the background field and density and the boundary conditions. Typical ionospheric boundary conditions would require that $\xi_{\beta r}$ be small at high latitudes, although b_γ may be large there. The combination of these factors will determine how effectively magnetic pulsations may be excited in the terrestrial magnetosphere, and the efficiency of heating by resonant absorption which is of interest in laboratory fusion devices and closed field line regions in the solar corona. It may be possible in some situations to adjust the density distribution and boundary conditions to give particularly efficient heating.

Most previous studies have concentrated upon the singular solution which dominates the perturbation around the resonance. To conclude we give the explicit leading order terms of our series solution. (The leading order compressional magnetic field $b_{\gamma 0}$ actually represents a combination of the regular and singular series.) Incorporating (4) into the series (25) yields

$$\xi_\alpha(\alpha,\gamma) \approx -i\beta_0 \left(\frac{k_\beta h_\alpha}{h_\beta}\right)_0 \xi_{\beta r}(0,\gamma)\ln(\alpha), \tag{45a}$$

$$\xi_\beta(\alpha,\gamma) \approx \frac{\beta_0}{\alpha}\, \xi_{\beta r}(0,\gamma), \tag{45b}$$

$$b_\gamma(\alpha,\gamma) \approx b_{\gamma 0}(\gamma). \tag{45c}$$

These relations confirm the α^{-1} and $\ln(\alpha)$ singularities found in the earliest one-dimensional uniform field models are still present in the more general resonances found in our two-dimensional partial differential equations. Our findings are contrary to those of HG, who claim that no such singularities will exist in systems like those considered in this paper. We find their calculation to be in error, and present a detailed discussion elsewhere.[25]

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APPENDIX A: PERMITTED INDEX VALUES

In this appendix we demonstrate that nontrivial solutions to (20) exist only when the index σ assumes the value of 0 or -1. The $\alpha^{\sigma-1}$ and $\alpha^{\sigma-1}\ln\alpha$ coefficients of (20b) yield

$$a_0\sigma + c_0 = 0, \tag{A1a}$$

$$\sigma c_0 = 0. \tag{A1b}$$

Suppose that $\sigma=0$: in this case (A1a) requires $c_0=0$. Alternatively, suppose that $\sigma\neq 0$: now (A1b) requires $c_0=0$. In either case we may conclude, for any value of σ, that (A1) may be written in the condensed form

$$c_0 = 0, \tag{A2a}$$

$$\sigma a_0 = 0. \tag{A2b}$$

The coefficients of the lowest order terms from (20a) contain no additional information, but the next order ($\alpha^{\sigma-1}$ and $\alpha^{\sigma-1}\ln\alpha$) yields the following relations:

$$a_1\sigma(\sigma+1) + c_1(2\sigma+1) = -ik_\beta(b_0\sigma + d_0), \tag{A3a}$$

$$c_1(\sigma+1)\sigma = -ik_\beta d_0\sigma, \tag{A3b}$$

whilst the next order of (20b) (α^σ and $\alpha^\sigma \ln\alpha$) yield

$$D_{\beta 0}(b_0) = -ik_\beta H^\gamma_{\alpha\beta 0}[a_1(\sigma+1) + c_1] + k_\beta^2 H^\gamma_{\alpha\beta 0} b_0, \tag{A4a}$$

$$D_{\beta 0}(d_0) = -ik_\beta H^\gamma_{\alpha\beta 0} c_1(\sigma+1) + k_\beta^2 H^\gamma_{\alpha\beta 0} d_0. \tag{A4b}$$

Multiplying (A3a) by σ, taking away (A3b), and then substituting for terms in a_1 and c_1 from (A4a) the result is

$$\sigma^2 D_{\beta 0}(b_0) = 0 \tag{A5a}$$

while substitution of (A3b) into (A4b) yields

$$\sigma D_{\beta 0}(d_0) = 0. \tag{A5b}$$

Moving on to the next order (α^σ and $\alpha^\sigma\ln\alpha$) of (20a) we find

$$\begin{aligned} &D_{\alpha 0}(a_0) + H^\gamma_{\alpha\beta 0}[a_2(\sigma+2)(\sigma+1) + c_2(2\sigma+3)] \\ &\quad + H^\gamma_{\alpha\beta 1}[a_1(\sigma+1)^2 + 2c_1(\sigma+1)] \\ &\quad = -ik_\beta\{H^\gamma_{\alpha\beta 0}[b_1(\sigma+1) + d_1] \\ &\qquad + H^\gamma_{\alpha\beta 1}[b_0(\sigma+1) + d_0]\}, \end{aligned} \tag{A6a}$$

$$\begin{aligned} &H^\gamma_{\alpha\beta 1} c_1(\sigma+1)^2 + H^\gamma_{\alpha\beta 0} c_2(\sigma+2)(\sigma+1) \\ &\quad = -ik_\beta[H^\gamma_{\alpha\beta 0} d_1(\sigma+1) + H^\gamma_{\alpha\beta 1} d_0(\sigma+1)] \end{aligned} \tag{A6b}$$

while the terms in $\alpha^{\sigma+1}$ and $\alpha^{\sigma+1}\ln\alpha$ from (20b) require

$$D_{\beta 0}(b_1)+D_{\beta 1}(b_0)$$
$$=-ik_\beta\{H^\gamma_{\alpha\beta 0}[a_2(\sigma+2)+c_2]$$
$$+H^\gamma_{\alpha\beta 1}[a_1(\sigma+1)+c_1]\}+k^2_\beta(H^\gamma_{\alpha\beta 0}b_1+H^\gamma_{\alpha\beta 1}b_0), \tag{A7a}$$

$$D_{\beta 0}(d_1)+D_{\beta 1}(d_0)$$
$$=-ik_\beta[H^\gamma_{\alpha\beta 0}c_2(\sigma+2)+H^\gamma_{\alpha\beta 1}c_1(\sigma+1)]$$
$$+k^2_\beta(H^\gamma_{\alpha\beta 0}d_1+H^\gamma_{\alpha\beta 1}d_0). \tag{A7b}$$

To summarize our conclusions so far [(A2) and (A5)], we find that for any value of σ the lowest order coefficients satisfy

$$\sigma a_0=\sigma^2 D_{\beta 0}(b_0)=c_0=\sigma D_{\beta 0}(d_0)=0. \tag{A8}$$

Now consider the case where σ has any value *except* 0 or -1. Under this restriction (A8) leads to the general solution

$$a_0=0,\quad b_0=\beta_0\tilde{\xi}_{\beta r},\quad c_0=0,\quad d_0=\delta_0\tilde{\xi}_{\beta r}, \tag{A9}$$

where $\tilde{\xi}_{\beta r}$ is the resonant eigenfunction satisfying $D_{\beta 0}(\tilde{\xi}_{\beta r})=0$, and β_0 and δ_0 are unspecified constants.

Multiplying (A7b) by $(\sigma+1)$ and subtracting (A6b) from the result we find (since we are assuming that $\sigma+1\neq 0$)

$$D_{\beta 0}(d_1)=-\delta_0 D_{\beta 1}\tilde{\xi}_{\beta r}, \tag{A10a}$$

where we have substituted d_0 from Eq. (A9). Similarly, we may multiply (A7a) by $(\sigma+1)$ and eliminate terms in a_2 and c_2 using (A6a) and (A7b), respectively. Employing (A9) and (A10a), and recalling that $\sigma\neq 0$, -1, it follows that

$$D_{\beta 0}(b_1)=-\beta_0 D_{\beta 1}\tilde{\xi}_{\beta r}. \tag{A10b}$$

To have well-behaved solutions it is necessary that we be able to invert the operator $D_{\beta 0}$ in (A10). In Sec. III, we deduced the solvability condition (15) that would guarantee the property of inversion. Applying this condition to (A10a) and (A10b) it follows that

$$\beta_0\langle\tilde{\xi}_{\beta r}D_{\beta 1}\tilde{\xi}_{\beta r}\rangle=\delta_0\langle\tilde{\xi}_{\beta r}D_{\beta 1}\tilde{\xi}_{\beta r}\rangle=0. \tag{A11}$$

The form of $\tilde{\xi}_{\beta r}$ is determined solely by the variation of the scale factors h_i and density along the resonant field line; $D_{\beta 0}(\tilde{\xi}_{\beta r})=0$—see (6). The operator $D_{\beta 1}$ depends upon how gradients of these quantities in α vary along the resonant field line, and is quite independent of $D_{\beta 0}$. (There may be some constraint upon the permissible variation of the scale factors, since B is both solenoidal and irrotational, however, the gradient of the plasma density and hence $D_{\beta 1}$ is quite arbitrary.)

Although it will certainly be possible to define a contrived a medium in which $\langle\tilde{\xi}_{\beta r}D_{\beta 1}\tilde{\xi}_{\beta r}\rangle=0$, a general medium will not meet this requirement. In general, the condition (A11) is satisfied by setting $\beta_0=\delta_0=0$. Consequently, all of the lowest order coefficients are zero [see (A9)], and we conclude that a solution with $\sigma\neq 0$, -1 would have $a_0=b_0=c_0=d_0$.

The calculation therefore leads us to the result we set out to prove: the only nontrivial solutions to (20) exist when $\sigma=-1$ or 0. This is comforting, since if we introduce simplifications into our model (e.g., let the magnetic field be uniform) we can recover the results of previous simple models. Indeed, in these models (governed by an ordinary differential equation) the solution generated with $\sigma=-1$ is the singular solution, and the one generated by assuming $\sigma=0$ is the regular solution (e.g., Ref. 34). In the main text it is shown how our method may be used to produce generalizations of their results.

APPENDIX B: GENERALIZED FROBENIUS COEFFICIENTS AND RECURRENCE RELATIONS

In this appendix we carry out the details of the calculation required to generate the regular and singular Frobenius series. To this end it is convenient to write general expressions for all of the terms from (20) of a given order. The coefficients of terms of order $\alpha^{\sigma+N}$ from (20a) satisfy ($N\geqslant 0$)

$$\sum_{i=0}^{N} D_{\alpha i}(a_{N-i})+\sum_{i=0}^{N+1}(i+1)H^\gamma_{\alpha\beta(i+1)}[a_{N+1-i}(\sigma+N+1$$
$$-i)+c_{N+1-i}]+\sum_{i=0}^{N+2}H^\gamma_{\alpha\beta i}[a_{N+2-i}(\sigma+N+2-i)$$
$$\times(\sigma+N+1-i)+c_{N+2-i}(2\sigma+2N+3-2i)]$$
$$=-ik_\beta\sum_{i=0}^{N}(i+1)H^\gamma_{\alpha\beta(i+1)}b_{N-i}$$
$$-ik_\beta\sum_{i=0}^{N+1}H^\gamma_{\alpha\beta i}[b_{N+1-i}(\sigma+N+1-i)+d_{N+1-i}] \tag{B1a}$$

while the $\alpha^{\sigma+N}\ln(\alpha)$ terms from (20a) yield

$$\sum_{i=0}^{N} D_{\alpha i}(c_{N-i})+\sum_{i=0}^{N+1}(i+1)H^\gamma_{\alpha\beta(i+1)}c_{N+1-i}$$
$$\times(\sigma+N+1-i)+\sum_{i=0}^{N+2}H^\gamma_{\alpha\beta i}c_{N+2-i}(\sigma+N+2-i)$$
$$\times(\sigma+N+1-i)$$
$$=-ik_\beta\sum_{i=0}^{N}(i+1)H^\gamma_{\alpha\beta(i+1)}d_{N-i}$$
$$-ik_\beta\sum_{i=0}^{N+1}H^\gamma_{\alpha\beta i}d_{N+1-i}(\sigma+N+1-i). \tag{B1b}$$

Similarly, the coefficients of the $\alpha^{\sigma+N}$ terms from (20b) yield

$$\sum_{i=0}^{N} D_{\beta i}(b_{N-i})$$
$$=-ik_\beta \sum_{i=0}^{N+1} H^\gamma_{\alpha\beta i}[a_{N+1-i}(\sigma+N+1-i)+c_{N+1-i}]$$
$$+k_\beta^2 \sum_{i=0}^{N} H^\gamma_{\alpha\beta i} b_{N-i} \qquad \text{(B2a)}$$

and the coefficients of $\alpha^{\sigma+N}\ln(\alpha)$ terms from (20b) satisfy

$$\sum_{i=0}^{N} D_{\beta i}(d_{N-i})=-ik_\beta \sum_{i=0}^{N+1} H^\gamma_{\alpha\beta i} c_{N+1-i}(\sigma+N+1-i)$$
$$+k_\beta^2 \sum_{i=0}^{N} H^\gamma_{\alpha\beta i} d_{N-i}. \qquad \text{(B2b)}$$

The relations above are valid for $N \geqslant 0$. Further equations that we need to supplement these general relations correspond to the $\alpha^{\sigma-1}$ and $\alpha^{\sigma-1}\ln(\alpha)$ coefficients of (20a) and of (20b). These are already given in (A3) and (A2), respectively. Equations (A2), (A3), (B1), and (B2) are a concise summary of all the relations required to solve for the coefficients. [Note that the $N=0$ versions of (B1a) and (B1b) have already been written explicitly as (A6a) and (A6b), respectively. The $N=1$ versions of (B2a) and (B2b) are also written explicitly as (A7a) and (A7b).]

1. The regular solution ($\sigma=0$)

Now we turn our attention to the generation of the regular solution and set σ equal to 0 in this subsection. Note that we are free to use the relations in (A2) here. For the regular series (A3a) reads

$$c_1=-ik_\beta d_0 \qquad \text{(B3a)}$$

while the $N=0$ relation from (B2b) reads

$$D_{\beta 0}(d_0)=-ik_\beta H^\gamma_{\alpha\beta 0}c_1+k_\beta^2 H^\gamma_{\alpha\beta 0}d_0. \qquad \text{(B3b)}$$

Combining (B3a) and (B3b) we find

$$D_{\beta 0}(d_0)=0; \quad \text{i.e., } d_0=\delta_0\tilde{\xi}_{\beta r}. \qquad \text{(B4)}$$

The constant δ_0 is unspecified at present, however, it is possible to place a constraint upon it by considering the next-order terms. The $N=0$ terms from (B1a) yield [after recalling the relations (A2)]

$$D_{\alpha 0}(a_0)+H^\gamma_{\alpha\beta 0}(2a_2+3c_2)+H^\gamma_{\alpha\beta 1}(a_1+2c_1)$$
$$=-ik_\beta H^\gamma_{\alpha\beta 1}b_0-ik_\beta H^\gamma_{\alpha\beta 0}(b_1+d_1)-ik_\beta H^\gamma_{\alpha\beta 1}d_0 \qquad \text{(B5a)}$$

while the $N=0$ terms from (B1b) require

$$H^\gamma_{\alpha\beta 1}c_1+2H^\gamma_{\alpha\beta 0}c_2=-ik_\beta H^\gamma_{\alpha\beta 0}d_1-ik_\beta H^\gamma_{\alpha\beta 1}d_0. \qquad \text{(B5b)}$$

The $N=1$ terms from (B2a) and (B2b) yield

$$D_{\beta 0}(b_1)+D_{\beta 1}(b_0)=-ik_\beta[H^\gamma_{\alpha\beta 0}(2a_2+c_2)+H^\gamma_{\alpha\beta 1}(a_1+c_1)]$$
$$+k_\beta^2[H^\gamma_{\alpha\beta 0}b_1+H^\gamma_{\alpha\beta 1}b_0] \qquad \text{(B6a)}$$

and

$$D_{\beta 0}(d_1)+D_{\beta 1}(d_0)=-ik_\beta(2H^\gamma_{\alpha\beta 0}c_2+H^\gamma_{\alpha\beta 1}c_1)$$
$$+k_\beta^2(H^\gamma_{\alpha\beta 0}d_1+H^\gamma_{\alpha\beta 1}d_0). \qquad \text{(B6b)}$$

Eliminating c_2 between (B5b) and (B6b) we find $D_{\beta 0}(d_1)+D_{\beta 1}(d_0)=0$, and applying the solvability condition (15) we require the constant δ_0 in (B4) to satisfy

$$\delta_0\langle\tilde{\xi}_{\beta r}D_{\beta 1}(\tilde{\xi}_{\beta r})\rangle=0. \qquad \text{(B7)}$$

Applying similar arguments to those following (A11), we conclude that in general the solution to (B7) is $\delta_0=0$, i.e., $d_0=0$. Consequently, c_1 is also zero by (B3a).

Of the lowest order coefficients we have deduced that $c_0=d_0=0$, from (A2) and (B7)—a reassuring result, since we would not anticipate any logarithmic functions in the regular solution. The other lowest order coefficients (a_0,b_0) are unconstrained and may be chosen to match some boundary conditions. There is, however, a general condition that a_0 and b_0 must satisfy and is found by eliminating a_2 from (B5a) and (B6a),

$$D_{\beta 0}(b_1)+D_{\beta 1}(b_0)=ik_\beta D_{\alpha 0}(a_0). \qquad \text{(B8a)}$$

Applying the solvability condition, we find

$$\langle\tilde{\xi}_{\beta r}[D_{\beta 1}(b_0)-ik_\beta D_{\alpha 0}(a_0)]\rangle=0. \qquad \text{(B8b)}$$

We now turn our attention to solving for the next coefficients in the series. [So far we have deduced that a_0 and b_0 are arbitrary, but satisfy (B8b), and $c_0=d_0=c_1=0$.] The coefficient a_1 may be determined from the $N=0$ solution of (B2a) which is already written explicitly in (A4a). For the regular solution, we find

$$a_1=\frac{i}{k_\beta H^\gamma_{\alpha\beta 0}}D_{\beta 0}(b_0)-ik_\beta b_0 \qquad \text{(B9a)}$$

while b_1 is found from inverting the Alfvén wave operator in (B8a)

$$b_1=b_1^{nr}+\beta_1\tilde{\xi}_{\beta r},$$
$$b_1^{nr}=\sum_{n\neq r}\frac{\langle\tilde{\xi}_{\beta n}(ik_\beta D_{\alpha 0}a_1-D_{\beta 1}b_0)\rangle}{(\omega^2-\omega_n^2)\langle\tilde{\xi}^2_{\beta n}(0,\gamma)G^{\beta\gamma}_{\alpha 0}\rangle}\tilde{\xi}_{\beta n}(0,\gamma). \qquad \text{(B9b)}$$

Here, b_1^{nr} is the nonresonant component of b_1, defined in accordance with (14), and β_1 is the amplitude of the resonant component (as yet undetermined).

The log terms play no role in the regular solution—a point to which we return later in this subsection. Consequently, the relations (B1b) and (B2b) provide no useful results (they are identically zero), however, the $N=1$ recursion relation from (B1a) and the $N=2$ relation from (B2a) provide the following useful information:

$$D_{\alpha 0}(a_1)+D_{\alpha 1}(a_0)+2H^\gamma_{\alpha\beta 1}a_2+2H^\gamma_{\alpha\beta 2}a_1$$
$$+6H^\gamma_{\alpha\beta 0}a_3+2H^\gamma_{\alpha\beta 1}a_2$$
$$=-ik_\beta H^\gamma_{\alpha\beta 1}b_1-2ik_\beta H^\gamma_{\alpha\beta 2}b_0-2ik_\beta H^\gamma_{\alpha\beta 0}b_2-ik_\beta H^\gamma_{\alpha\beta 1}b_1 \qquad \text{(B10a)}$$

and

$$D_{\beta 0}(b_2)+D_{\beta 1}(b_1)+D_{\beta 2}(b_0)$$
$$=-3ik_\beta H^\gamma_{\alpha\beta 0}a_3-2ik_\beta H^\gamma_{\alpha\beta 1}a_2-ik_\beta H^\gamma_{\alpha\beta 2}a_1+k_\beta^2 H^\gamma_{\alpha\beta 0}b_2 +k_\beta^2 H^\gamma_{\alpha\beta 1}b_1+k_\beta^2 H^\gamma_{\alpha\beta 2}b_0. \tag{B10b}$$

Eliminating a_3 between (B10a) and (B10b) we find

$$D_{\beta 0}(b_2)+D_{\beta 1}(b_1)+D_{\beta 2}(b_0) =(ik_\beta/2)[D_{\alpha 0}(a_1)+D_{\alpha 1}(a_0)]. \tag{B11}$$

Applying the solvability condition to (B11), we may deduce the value of the constant β_1 in (B9b),

$$\beta_1=\frac{\langle\tilde{\xi}_{\beta r}[(ik_\beta/2)[D_{\alpha 0}(a_1)+D_{\alpha 1}(a_0)]-D_{\beta 1}(b_1^{nr})-D_{\beta 2}(b_0)]\rangle}{\langle\tilde{\xi}_{\beta r}D_{\beta 1}(\tilde{\xi}_{\beta r})\rangle}. \tag{B12}$$

The coefficient b_2 may be determined, up to a component of the resonant eigenfunction $(\beta_2\tilde{\xi}_{\beta r})$, by inverting the Alfvén wave operator in (B11),

$$b_2=b_2^{nr}+\beta_2\tilde{\xi}_{\beta r}, \tag{B13a}$$

$$b_2^{nr}=\sum_{n\neq r}\frac{\langle\tilde{\xi}_{\beta n}[(ik_\beta/2)[D_{\alpha 0}(a_1)+D_{\alpha 1}(a_0)]-D_{\beta 1}(b_1)-D_{\beta 2}(b_0)]\rangle}{(\omega^2-\omega_n^2)\langle\tilde{\xi}^2_{\beta n}(0,\gamma)G^{\beta\gamma}_{\alpha 0}\rangle}\tilde{\xi}_{\beta n}(0,\gamma), \tag{B13b}$$

while a_2 may be found from either the $N=0$ relation of (B1a) or the $N=1$ relation of (B2a). For example, employing the former recursion relation we find

$$a_2=\frac{1}{2H^\gamma_{\alpha\beta 0}}[-ik_\beta(H^\gamma_{\alpha\beta 1}b_0+H^\gamma_{\alpha\beta 0}b_1)-D_{\alpha 0}(a_0) -H^\gamma_{\alpha\beta 1}a_1]. \tag{B14}$$

If the relation from (B2a) had been used, it is possible to prove that the two results are equivalent by realizing that (8) and (B9b) yield the property

$$D_{\beta 0}(b_1)=\sum_{n\neq r}\frac{\langle\tilde{\xi}_{\beta n}(ik_\beta D_{\alpha 0}a_1-D_{\beta 1}b_0)\rangle}{\langle\tilde{\xi}^2_{\beta n}(0,\gamma)G^{\beta\gamma}_{\alpha 0}\rangle}G^{\beta\gamma}_{\alpha 0}\tilde{\xi}_{\beta n}(0,\gamma). \tag{B15}$$

The general procedure for constructing the regular solution is now becoming evident. Suppose we know all of the a_n and b_n up to and including $n=m$, except for the constant β_m. We begin by taking the $N=m$ and $N=m+1$ recursion relations from (B1a) and (B2a), respectively. Eliminating $a_{(m+2)}$ from these relations yields an equation of the form

$$D_{\beta 0}(b_{(m+1)})+D_{\beta 1}(b_m)+D_{\beta 2}(b_{(m-1)})+\cdots=\cdots. \tag{B16}$$

Applying the solvability condition (15) to (B16) determines the constant β_m, while inverting the Alfvén wave operator in (B16) determines the nonresonant component $(b^{nr}_{(m+1)})$ of $b_{(m+1)}$. The resonant component $\beta_{(m+1)}\tilde{\xi}_{\beta r}$ is determined when this procedure is repeated. The coefficient $a_{(m+1)}$ may be found from either the $N=m-2$ and $N=m-1$ recursion relations of (B1a) and (B2a), respectively.

We have explicitly determined the first three coefficients in the regular series, except for the constant β_2. We leave it as an exercise to the interested reader to show that the above prescription yields

$$\beta_2=\frac{\langle\tilde{\xi}_{\beta r}[(ik_\beta/3)[D_{\alpha 0}(a_2)+D_{\alpha 1}(a_1)+D_{\alpha 2}(a_0)]-D_{\beta 1}(b_2^{nr})-D_{\beta 2}(b_1)-D_{\beta 3}(b_0)]\rangle}{\langle\tilde{\xi}_{\beta r}D_{\beta 1}(\tilde{\xi}_{\beta r})\rangle}. \tag{B17}$$

We proved earlier that the logarithmic coefficients c_0, d_0, and c_1 were all zero in the regular solution. We now prove by induction that all of the c_n and d_n coefficients are zero.

Suppose that all c_n and d_n are zero for $n=0, 1, 2,...,m$. Then (B2b) with $N=m$ reads

$$0=-ik_\beta(m+1)H^\gamma_{\alpha\beta 0}c_{(m+1)} \tag{B18a}$$

while the $N=m+1$ relation of (B2b) is

$$D_{\beta 0}(d_{(m+1)})=-ik_\beta(m+2)H^\gamma_{\alpha\beta 0}c_{(m+2)} +k_\beta^2 H^\gamma_{\alpha\beta 0}d_{(m+1)}. \tag{B18b}$$

Similarly, (B1b) with $N=m$ yields

$$(m+2)c_{(m+2)}=-ik_\beta d_{(m+1)}. \tag{B18c}$$

Evidently, Eq. (B18a) proves that the next coefficient in the series $c_{(m+1)}$ will be zero. Eliminating $c_{(m+2)}$ between (B18b) and (B18c) we conclude that $d_{(m+1)}$ must be proportional to the resonant eigenfunction,

$$D_{\beta 0}(d_{(m+1)})=0, \quad d_{(m+1)}=\delta_{(m+1)}\tilde{\xi}_{\beta r}. \tag{B18d}$$

The constant $\delta_{(m+1)}$ is determined by applying the solvability condition to the next order of the equations: Eliminating $c_{(m+3)}$ between the $N=m+1$ relation of (B1b) and the $N=m+2$ relation of (B2b), we find

$$D_{\beta 0}(d_{(m+2)})+D_{\beta 1}(\delta_{(m+1)}\tilde{\xi}_{\beta r})=0. \tag{B18e}$$

Once again the solvability condition requires that $\delta_{(m+1)}$ [and consequently $(d_{(m+1)})$] be equal to zero. Since we already know that d_0 and c_0 are zero in the regular solution, it follows by induction that all of the higher c_n and d_n are also zero. Thus there are no logarithmic terms in the regular ($\sigma=0$) Frobenius series.

2. The singular solution ($\sigma=-1$)

In this subsection we consider how to construct the singular Frobenius series, and σ will always adopt the value of -1. The general results (A8) give the following lowest order coefficients

$$a_0=0, \quad b_0=\beta_0\tilde{\xi}_{\beta r}, \quad c_0=0, \quad d_0=\delta_0\tilde{\xi}_{\beta r}. \tag{B19}$$

The lowest order equations (A3), (A4), (A6), and (A7) may also be employed: (A3b) reads $0=ik_\beta d_0$, while (A3a) gives the value of c_1,

$$d_0=0, \quad c_1=-ik_\beta\beta_0\tilde{\xi}_{\beta r}. \tag{B20a}$$

For the singular solution equation (A6a) reads

$$c_2=-ik_\beta d_1. \tag{B20b}$$

Employing the results (B20a) and (B20b), relation (A7b) simplifies to

$$D_{\beta 0}(d_1)=0; \quad d_1=\delta_1\tilde{\xi}_{\beta r}. \tag{B20c}$$

The constant δ_1 will be determined later. The other recursion relation from this order is (A7a); incorporating the above properties we find the relation becomes

$$D_{\beta 0}(b_1)+\beta_0 D_{\beta 1}(\tilde{\xi}_{\beta r})=-ik_\beta H^\gamma_{\alpha\beta 0}(a_2-ik_\beta d_1)+k_\beta^2 H^\gamma_{\alpha\beta 0}b_1 \tag{B20d}$$

and will be used presently to determine a_2.

To proceed further we need to consider the higher order recursion relations. The $N=1$ relations from (B1a) and (B1b) supply the equations

$$D_{\alpha 0}(a_1)+H^\gamma_{\alpha\beta 1}(a_2-ik_\beta\delta_1\tilde{\xi}_{\beta r})+H^\gamma_{\alpha\beta 0}(2a_3+3c_3)=-ik_\beta H^\gamma_{\alpha\beta 1}b_1-ik_\beta H^\gamma_{\alpha\beta 0}(b_2+d_2) \tag{B21a}$$

and

$$D_{\alpha 0}(c_1)+2H^\gamma_{\alpha\beta 0}c_3=-ik_\beta H^\gamma_{\alpha\beta 0}d_2. \tag{B21b}$$

While the $N=2$ relations of (B2a) and (B2b) furnish the recursion properties

$$D_{\beta 0}(b_2)+D_{\beta 1}(b_1)+\beta_0 D_{\beta 2}(\tilde{\xi}_{\beta r})=-ik_\beta H^\gamma_{\alpha\beta 0}(2a_3+c_3)-ik_\beta H^\gamma_{\alpha\beta 1}(a_2-ik_\beta\delta_1\tilde{\xi}_{\beta r})+k_\beta^2 H^\gamma_{\alpha\beta 0}b_2+k_\beta^2 H^\gamma_{\alpha\beta 1}b_1 \tag{B22a}$$

and

$$D_{\beta 0}(d_2)+\delta_1 D_{\beta 1}(\tilde{\xi}_{\beta r})=-2ik_\beta H^\gamma_{\alpha\beta 0}c_3+k_\beta^2 H^\gamma_{\alpha\beta 0}d_2, \tag{B22b}$$

respectively. [In the above we have employed (B19) and (B20).]

Eliminating c_3 between the relations (B21b) and (B22b) yields

$$D_{\beta 0}(d_2)+\delta_1 D_{\beta 1}(\tilde{\xi}_{\beta r})=k_\beta^2\beta_0 D_{\alpha 0}(\tilde{\xi}_{\beta r}) \tag{B23a}$$

while eliminating a_3 between (B21a) and (B22a) gives

$$D_{\beta 0}(b_2)+D_{\beta 1}(b_1)+\beta_0 D_{\beta 2}(\tilde{\xi}_{\beta r})+D_{\beta 0}(d_2)+\delta_1 D_{\beta 1}(\tilde{\xi}_{\beta r})-ik_\beta D_{\alpha 0}(a_1)=0. \tag{B23b}$$

We can determine δ_1 (i.e., d_1) by applying the solvability condition to (B23a)

$$\delta_1=k_\beta^2\beta_0\frac{\langle\tilde{\xi}_{\beta r}D_{\alpha 0}(\tilde{\xi}_{\beta r})\rangle}{\langle\tilde{\xi}_{\beta r}D_{\beta 1}(\tilde{\xi}_{\beta r})\rangle}. \tag{B24a}$$

Note that in the uniform field calculation, which many studies have focused upon, the operator $D_{\alpha 0}$ is identical to $D_{\beta 0}$. Consequently, in such studies the RHS of (B23a) would be zero, as would d_1.

Applying the solvability condition to (B23b), we find

$$\langle\tilde{\xi}_{\beta r}[D_{\beta 1}(b_1)-ik_\beta D_{\alpha 0}(a_1)]\rangle=-\langle\tilde{\xi}_{\beta r}[\beta_0 D_{\beta 2}(\tilde{\xi}_{\beta r})+\delta_1 D_{\beta 1}(\tilde{\xi}_{\beta r})]\rangle. \tag{B24b}$$

Apart from this constraint upon (a_1,b_1) the coefficients are arbitrary. Indeed, we could modify a_1 by Δa_1 and b_1 by Δb_1 and still satisfy (B24b) provided that

$$\langle\tilde{\xi}_{\beta r}[D_{\beta 1}(\Delta b_1)-ik_\beta D_{\alpha 0}(\Delta a_1)]\rangle=0. \tag{B25}$$

It is interesting to compare this relation with the condition we found the regular solution must satisfy, namely (B8b). The two equations are identical save for the indices of the coefficients. Note that in the singular solution ($\sigma=-1$) a_1 and b_1 describe displacements of order α^0, similarly in the regular solution ($\sigma=0$) a_0 and b_0 also correspond to the order α^0. It is evident that modifying the singular solution by Δa_1 and Δb_1 is equivalent to adding on a component of the regular solution. For the remainder of this section we shall, without loss of generality, set the singular coefficients a_1 and b_1 to be

$$a_1=0, \quad b_1=\beta_1\tilde{\xi}_{\beta r}, \tag{B26}$$

since any physical solution may be constructed by adding on an appropriate regular solution later.

Given the singular coefficients in (B26) we may determine the value of β_1 directly from substitution into (B24b)

$$\beta_1=-\frac{\langle\tilde{\xi}_{\beta r}[\beta_0 D_{\beta 2}(\tilde{\xi}_{\beta r})+\delta_1 D_{\beta 1}(\tilde{\xi}_{\beta r})]\rangle}{\langle\tilde{\xi}_{\beta r}D_{\beta 1}(\tilde{\xi}_{\beta r})\rangle}. \tag{B27}$$

So far we have determined (a_0,b_0,c_0,d_0) and (a_1,b_1,c_1,d_1) for the singular solution. The coefficients a_2 and c_2 follow directly from (B20d) and (B20b), respectively,

$$a_2=i\frac{\beta_0}{k_\beta H^{\gamma}_{\alpha\beta0}}D_{\beta 1}(\tilde{\xi}_{\beta r})+ik_\beta\delta_1\tilde{\xi}_{\beta r}-ik_\beta\beta_1\tilde{\xi}_{\beta r}, \tag{B28a}$$

$$c_2=-ik_\beta^3\beta_0\frac{\langle\tilde{\xi}_{\beta r}D_{\alpha 0}(\tilde{\xi}_{\beta r})\rangle}{\langle\tilde{\xi}_{\beta r}D_{\beta 1}(\tilde{\xi}_{\beta r})\rangle}\tilde{\xi}_{\beta r}. \tag{B28b}$$

The nonresonant components of the coefficients b_2 and d_2 may be determined from inversion of the wave operator $D_{\beta 0}$ in Eqs. (B23b) and (B23a), respectively,

$$b_2=b_2^{nr}+\beta_2\tilde{\xi}_{\beta r}, \tag{B28c}$$

$$b_2^{nr}=-\sum_{n\neq r}\frac{\langle\tilde{\xi}_{\beta n}[(\beta_1+\delta_1)D_{\beta 1}\tilde{\xi}_{\beta r}+\beta_0 D_{\beta 2}\tilde{\xi}_{\beta r}+D_{\beta 0}d_2^{nr}]\rangle}{(\omega^2-\omega_n^2)\langle\tilde{\xi}_{\beta n}^2(0,\gamma)G^{\beta\gamma}_{\alpha 0}\rangle}\times\tilde{\xi}_{\beta n}(0,\gamma), \tag{B28d}$$

$$d_2=d_2^{nr}+\delta_2\tilde{\xi}_{\beta r}, \tag{B28e}$$

$$d_2^{nr}=\sum_{n\neq r}\frac{\langle\tilde{\xi}_{\beta n}[k_\beta^2\beta_0 D_{\alpha 0}\tilde{\xi}_{\beta r}-\delta_1 D_{\beta 1}\tilde{\xi}_{\beta r}]\rangle}{(\omega^2-\omega_n^2)\langle\tilde{\xi}_{\beta n}^2(0,\gamma)G^{\beta\gamma}_{\alpha 0}\rangle}\tilde{\xi}_{\beta n}(0,\gamma), \tag{B28f}$$

the constants β_2 and δ_2 to be determined later.

An algorithm is now becoming apparent by which we may construct the series of coefficients. For example, suppose we know all of the coefficients up to $(a_m,b_{(m-1)},c_m,d_{(m-1)})$. Furthermore, let us know the nonresonant components b_m^{nr} and d_m^{nr} of the coefficients b_m and d_m, but not their resonant components β_m and δ_m: $(b_m=b_m^{nr}+\beta_m\tilde{\xi}_{\beta r};\ d_m=d_m^{nr}+\delta_m\tilde{\xi}_{\beta r})$. How do we calculate the next order of coefficients?

Calculating the next orders of the b and d coefficients is fairly complicated, and involves inverting the Alfvén wave operator $D_{\beta 0}$. However, it is relatively easy to calculate the next coefficients for a and c. The following four steps will advance our current knowledge to the next order. (1) The $N=m-1$ version of (B1a) gives the term $c_{(m+1)}$. (2) The $N=m$ version of (B2a) yields $a_{(m+1)}$. (3) Eliminate $c_{(m+2)}$ between the $N=m$ version of (B1b) and the $N=m+1$ version of (B2b) to get an equation of the form

$$D_{\beta 0}(d_{(m+1)})+D_{\beta 1}(d_m)+\cdots=\cdots. \tag{B29a}$$

Applying the solvability condition to the above relation will determine the constant δ_m, whereas inverting the operator $D_{\beta 0}$ will yield the nonresonant component $(d^{nr}_{(m+1)})$ of $d_{(m+1)}$. (4) Finally, we take the $N=m$ edition of (B1a), use the $N=m+1$ version of (B2a) to eliminate $a_{(m+2)}$ and then the $N=m+1$ edition of (B2b) to remove $c_{(m+2)}$. The result is something of the form

$$D_{\beta 0}(b_{(m+1)})+D_{\beta 1}(b_m)+\cdots+D_{\beta 0}(d_{(m+1)})+\cdots=0. \tag{B29b}$$

[All of the terms represented by $\cdots$ are known quantities. Note that the undetermined resonant component $(\delta_{(m+1)}\tilde{\xi}_{\beta r})$ does not affect (B29b) since it is operated on by $D_{\beta 0}$, giving zero.] Applying the solvability condition to (B29b) will determine the constant β_m, while inverting the operator $D_{\beta 0}$ will determine the nonresonant component $(b^{nr}_{(m+1)})$ of $b_{(m+1)}$. We have now advanced to the next order. This process may be repeated indefinitely, and as many terms as are desired in the Generalized Frobenius Series may be determined.

We leave it as an exercise to the interested reader to confirm that the $m=2$ editions of (B29a) and (B29b) are

$$D_{\beta 0}(d_3)+D_{\beta 1}(d_2)+D_{\beta 2}(d_1)=(ik_\beta/2)[D_{\alpha 0}(c_2)+D_{\alpha 1}(c_1)] \tag{B30a}$$

and

$$D_{\beta 0}(b_3)+D_{\beta 1}(b_2)+D_{\beta 2}(b_1)+D_{\beta 3}(b_0)+\tfrac{1}{2}[D_{\beta 0}(d_3)+D_{\beta 1}(d_2)+D_{\beta 2}(d_1)-ik_\beta D_{\alpha 0}(a_2)]=0. \tag{B30b}$$

Applying the solvability condition to the above equations determines β_2 and δ_2,

$$\beta_2=\frac{\langle\tilde{\xi}_{\beta r}[\tfrac{1}{2}(ik_\beta D_{\alpha 0}a_2-D_{\beta 1}d_2^{nr}-\delta_2 D_{\beta 1}\tilde{\xi}_{\beta r}-\delta_1 D_{\beta 2}\tilde{\xi}_{\beta r})-D_{\beta 1}b_2^{nr}-\beta_1 D_{\beta 2}\tilde{\xi}_{\beta r}-\beta_0 D_{\beta 3}\tilde{\xi}_{\beta r}]\rangle}{\langle\tilde{\xi}_{\beta r}D_{\beta 1}(\tilde{\xi}_{\beta r})\rangle}, \tag{B31a}$$

$$\delta_2=\frac{\langle\tilde{\xi}_{\beta r}[(k_\beta^2/2)[\delta_1 D_{\alpha 0}\tilde{\xi}_{\beta r}+\beta_0 D_{\alpha 1}\tilde{\xi}_{\beta r}]-D_{\beta 1}d_2^{nr}-\delta_1 D_{\beta 2}\tilde{\xi}_{\beta r}]\rangle}{\langle\tilde{\xi}_{\beta r}D_{\beta 1}(\tilde{\xi}_{\beta r})\rangle}. \tag{B31b}$$

[Equations (B20a) and (B20b) were employed in evaluating the above expressions.] Thus we have derived the first three orders of the coefficients in both the regular and singular Generalized Frobenius Series.

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